Investigating a new estimator of the serial correlation coefficient

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8 May 1998

Abstract

A new estimator of the lag-1 serial correlation coefficient ρ, $\sum_{i=1}^{n-1} \epsilon_i^3 \epsilon_{i+1} \ / \ \sum_{i=1}^{n-1} \epsilon_i^4$, is compared with the old estimator, $\sum_{i=1}^{n-1} \epsilon_i \epsilon_{i+1} \ / \ \sum_{i=1}^{n-1} \epsilon_i^2$, for a stationary AR(1) process with known mean. The mean and the variance of both estimators are calculated using the second order Taylor expansion of a ratio. No further approximation is used. In case of the mean of the old estimator, we derive Marriott and Pope's (1954) formula, with $(n-1)^{-1}$ instead of $(n)^{-1}$, and an additional term $\propto (n-1)^{-2}$. In case of the variance of the old estimator, Bartlett's (1946) formula results, with $(n-1)^{-1}$ instead of $(n)^{-1}$. The theoretical expressions are corroborated with simulation experiments. The main results are as follows. (1) The new estimator has a larger negative bias and a larger variance than the old estimator. (2) The theoretical results for the mean and the variance of the old estimator describe the principal behaviours over the entire range of ρ, in particular, the decline to zero negative bias as ρ approaches unity.

Mudelsee M (1998) Investigating a new estimator of the serial correlation coefficient. Institute of Mathematics and Statistics, University of Kent, Canterbury, IMS Technical Report UKC/IMS/98/15, Canterbury, United Kingdom, 22 pp.

1 Introduction

Consider the following estimators of the serial correlation coefficient ρ of lag 1 from a time series ϵ_i $(i = 1, \ldots, n)$ sampled from a process $\mathcal{E}_i$ with known (zero) mean:

$$\hat{\rho} = \frac{\sum_{i=1}^{n-1} \epsilon_i \epsilon_{i+1}}{\sum_{i=1}^{n-1} \epsilon_i^2}, \tag{1}$$

$$\hat{\rho}^\star = \frac{\sum_{i=1}^{n-1} \epsilon_i^3 \epsilon_{i+1}}{\sum_{i=1}^{n-1} \epsilon_i^4}. \tag{2}$$

Estimators of type (1) are well-known (e. g. Bartlett 1946, Marriott and Pope 1954), whereas (2) is new to my best knowledge.

Let $\mathcal{E}_i$ be the stationary AR(1) process,

$$\begin{aligned} \mathcal{E}_1 &\sim N(0,\ 1), \\ \mathcal{E}_i &= \rho\, \mathcal{E}_{i-1} + \mathcal{U}_i, \quad i = 2, \ldots, n, \end{aligned} \tag{3}$$

with $\mathcal{U}_i$ i. i. d. $\sim N(0,\ \sigma^2)$ and $\sigma^2 = 1 - \rho^2$. The following misjudgement of mine led to the present study. $\hat{\rho}^\star$ minimises

$$\sum_{i=1}^{n-1} (\epsilon_{i+1} - \rho \epsilon_i)^2\, \epsilon_i^2,$$

which is a weighted sum of squares. However, since $\mathcal{E}_i$ has constant variance unity, no weighting should be necessary.

Nevertheless, we compare estimators (1) and (2) for process (3). We restrict ourselves to $0 < \rho < 1$. We first calculate their variances, respectively means, up to the second order of the Taylor expansion of a ratio, similarly to Bartlett (1946), respectively Marriott and Pope (1954). Since we concentrate on the lag-1 estimators and process (3), no further approximation is necessary. These theoretical expressions and also those of White (1961) for estimator (1) are then examined with simulation experiments.

2 Series expansions

2.1 Old estimator

2.1.1 Variance

We write (1) as

$$\begin{aligned} \hat{\rho} &= \frac{\frac{1}{n}\sum_{i=1}^{n-1} \epsilon_i \epsilon_{i+1}}{\frac{1}{n}\sum_{i=1}^{n-1} \epsilon_i^2} \qquad (4) \\ &= \frac{c}{v}, \text{ say,} \end{aligned}$$

and have, up to the second order of deviations from the means of c and v, the variance

$$\mathrm{var}(\hat{\rho}) = \mathrm{var}(c/v) \simeq \frac{\mathrm{var}(c)}{\mathrm{E}^2(v)} - 2\,\mathrm{E}(c)\frac{\mathrm{cov}(v,c)}{\mathrm{E}^3(v)} + \mathrm{E}^2(c)\frac{\mathrm{var}(v)}{\mathrm{E}^4(v)}. \tag{5}$$

We now apply a standard result for quadravariate standard Gaussian distributions with serial correlations ρ_j (e. g. Priestley 1981:325),

$$\mathrm{cov}(\epsilon_a\epsilon_{a+s}, \epsilon_b\epsilon_{b+s+t}) = \rho_{b-a}\,\rho_{b-a+t} \;+\; \rho_{b-a+s+t}\,\rho_{b-a-s},$$

from which we derive, for ϵ_i following process (3):

$$\mathrm{cov}\left(\frac{1}{n}\sum_{a=1}^{n-1}\epsilon_a\epsilon_{a+s}, \frac{1}{n}\sum_{b=1}^{n-1}\epsilon_b\epsilon_{b+s+t}\right) =$$

$$= \frac{1}{n^2}\sum_{a,b=1}^{n-1}(\rho^{|b-a|}\,\rho^{|b-a+t|} \;+\; \rho^{|b-a+s+t|}\,\rho^{|b-a-s|}). \tag{6}$$

Without further approximation we can directly derive the various constituents of the right-hand side of (5) by the means of (6).

$$\begin{aligned}\mathrm{var}(c) &= \mathrm{var}\left(\frac{1}{n}\sum_{i=1}^{n-1}\epsilon_i\epsilon_{i+1}\right) = (\text{put } s=1 \text{ and } t=0 \text{ in } (6))\\ &= \frac{1}{n^2}\left(\sum_{a,b=1}^{n-1}\rho^{2|b-a|} + \sum_{a,b=1}^{n-1}\rho^{|b-a+1|}\,\rho^{|b-a-1|}\right).\end{aligned}$$

We find, by summing over the points of the a-b lattice in a 'diagonal' manner,

$$\begin{aligned}\sum_{a,b=1}^{n-1}\rho^{2|b-a|} &= (n-1) + 2\sum_{i=1}^{n-2} i\,\rho^{2(n-i-1)}\\ &= (n-1)\left(\frac{2}{1-\rho^2} - 1\right) + 2\,\frac{(\rho^{2n-4} - \rho^{-2})}{(1-\rho^{-2})^2}\end{aligned} \tag{7}$$

(at the last step we have used the arithmetic-geometric progression), and

$$\begin{aligned}\sum_{a,b=1}^{n-1}\rho^{|b-a+1|}\,\rho^{|b-a-1|} &= (n-1)\,\rho^2 + 2\sum_{i=1}^{n-2} i\,\rho^{2(n-i-1)}\\ &= (n-1)\left(\frac{2}{1-\rho^2} + \rho^2 - 2\right) + 2\,\frac{(\rho^{2n-4} - \rho^{-2})}{(1-\rho^{-2})^2}.\end{aligned} \tag{8}$$

(7) and (8) give var(c). Further,

$$\mathrm{cov}(v,c) = \mathrm{cov}\left(\frac{1}{n}\sum_{i=1}^{n-1}\epsilon_i^2, \frac{1}{n}\sum_{i=1}^{n-1}\epsilon_i\epsilon_{i+1}\right)$$

$$
\begin{aligned}
&= (\text{put } s=0 \text{ and } t=1 \text{ in } (6)) \\
&= \frac{2}{n^2} \sum_{a,b=1}^{n-1} \rho^{|b-a|}\, \rho^{|b-a+1|}.
\end{aligned}
$$

We find

$$
\begin{aligned}
\sum_{a,b=1}^{n-1} \rho^{|b-a|}\, \rho^{|b-a+1|} &= \rho^{2n-3} + \sum_{i=1}^{n-2} (2\,i+1)\, \rho^{2(n-i)-3} \\
&= (n-1)\,\frac{2}{\rho}\left(\frac{1}{1-\rho^2} - 1\right) \\
&\quad +2\,\frac{(\rho^{2n-5}-\rho^{-3})}{(1-\rho^{-2})^2} + \frac{(\rho^{2n-3}-\rho^{-1})}{(1-\rho^{-2})}. \qquad (9)
\end{aligned}
$$

(At the last step we have used the arithmetic-geometric and also the geometric progression.) (9) gives $\mathrm{cov}(v,c)$. Further,

$$
\begin{aligned}
\mathrm{var}(v) &= \mathrm{var}\left(\frac{1}{n}\sum_{i=1}^{n-1} \epsilon_i^2\right) = (\text{put } s=t=0 \text{ in } (6)) \\
&= \frac{2}{n^2} \sum_{a,b=1}^{n-1} \rho^{2|b-a|},
\end{aligned}
$$

which is given from (7). We further need

$$
\mathrm{E}(v) = \frac{n-1}{n} \qquad (10)
$$

and

$$
\mathrm{E}(c) = \frac{\rho(n-1)}{n}. \qquad (11)
$$

All three terms contributing to $\mathrm{var}(\hat{\rho})$ have a part $\propto (n-1)^{-2}$. However, these parts cancel out:

$$
\mathrm{var}(\hat{\rho}) \simeq \frac{(1-\rho^2)}{(n-1)}. \qquad (12)
$$

Bartlett (1946) already gave, up to the order $(n)^{-1}$,

$$
\mathrm{var}(\hat{\rho}) \simeq \frac{(1-\rho^2)}{n}, \qquad (13)
$$

which cannot be distinguished from our result.

2.1.2 Mean

We have, up to the second order of deviations from the means of c and v, the mean

$$\mathrm{E}(\hat{\rho}) = \mathrm{E}(c/v) \simeq \frac{\mathrm{E}(c)}{\mathrm{E}(v)} - \frac{\mathrm{cov}(v,c)}{\mathrm{E}^2(v)} + \mathrm{E}(c)\frac{\mathrm{var}(v)}{\mathrm{E}^3(v)}. \tag{14}$$

As above, we derive $\mathrm{cov}(v,c)$ and $\mathrm{var}(v)$ from (9) and (7), respectively, and have $\mathrm{E}(v)$ and $\mathrm{E}(c)$ given by (10) and (11), respectively, yielding

$$\mathrm{E}(\hat{\rho}) \simeq \rho - \frac{2\,\rho}{(n-1)} + \frac{2}{(n-1)^2}\frac{(\rho-\rho^{2n-1})}{(1-\rho^2)}. \tag{15}$$

Marriott and Pope (1954) investigated the estimator

$$\hat{\rho}' \quad = \quad \frac{\frac{1}{n-1}\sum_{i=1}^{n-1}\epsilon_i\epsilon_{i+1}}{\frac{1}{n}\sum_{i=1}^{n-1}\epsilon_i^2}$$

and gave, up to the order $(n)^{-1}$,

$$\mathrm{E}(\hat{\rho}') \simeq \rho - \frac{2\,\rho}{n},$$

which cannot be distinguished from the first two terms of our result.

2.1.3 Unknown mean of the process

I have also tried to calculate up to the same order of approximation the mean and the variance of the estimator

$$\frac{\sum_{i=1}^{n-1}\left(\epsilon_i - \frac{1}{n-1}\sum_{j=1}^{n-1}\epsilon_j\right)\left(\epsilon_{i+1} - \frac{1}{n-1}\sum_{j=1}^{n-1}\epsilon_{j+1}\right)}{\sum_{i=1}^{n-1}\left(\epsilon_i - \frac{1}{n-1}\sum_{j=1}^{n-1}\epsilon_j\right)^2},$$

for the case when the mean of the process is unknown. That would require to calculate the following sums over the points of a cubic lattice:

$$\sum_{a,b,c=1}^{n-1}\rho^{|b-a|}\,\rho^{|c-a-s|}$$

for $s = 0$ and 1. However, I was unable to obtain exact formulas.

2.2 New estimator

2.2.1 Variance

We write (2) as

$$\hat{\rho}^\star = \frac{\frac{1}{n}\sum_{i=1}^{n-1}\epsilon_i^3\epsilon_{i+1}}{\frac{1}{n}\sum_{i=1}^{n-1}\epsilon_i^4} = \frac{d}{w}, \text{ say.}$$

$\mathrm{var}(\hat{\rho}^\star)$, up to the second order of the Taylor expansion, is given by (5), substituting d for c and w for v, respectively. We need the following result for octavariate standard Gaussian distributions with serial correlations ρ_j (Appendix),

$$\begin{aligned}
\mathrm{cov}(\epsilon_a^3\epsilon_{a+s}, \epsilon_b^3\epsilon_{b+s+t}) \;=\;& 9\,\rho_{b-a}\,\rho_{b-a+t} \;+\; 9\,\rho_{b-a+s+t}\,\rho_{b-a-s} \\
&+ 18\,\rho_s\,\rho_{b-a}\,\rho_{b-a+s+t} \;+\; 18\,\rho_{b-a}\,\rho_{b-a-s}\,\rho_{s+t} \\
&+ 18\,\rho_s\,\rho_{b-a}^2\,\rho_{s+t} \;+\; 18\,\rho_{b-a}^2\,\rho_{b-a+s+t}\,\rho_{b-a-s} \\
&+ 6\,\rho_{b-a}^3\,\rho_{b-a+t},
\end{aligned} \tag{16}$$

from which we derive, for ϵ_i following process (3):

$$\begin{aligned}
&\mathrm{cov}\left(\frac{1}{n}\sum_{a=1}^{n-1}\epsilon_a^3\epsilon_{a+s}, \frac{1}{n}\sum_{b=1}^{n-1}\epsilon_b^3\epsilon_{b+s+t}\right) = \\
&\quad = \frac{1}{n^2}\sum_{a,b=1}^{n-1}\big(\, 9\,\rho^{|b-a|}\,\rho^{|b-a+t|} \;+\; 9\,\rho^{|b-a+s+t|}\,\rho^{|b-a-s|} \\
&\qquad + 18\,\rho^{|s|}\,\rho^{|b-a|}\,\rho^{|b-a+s+t|} \;+\; 18\,\rho^{|b-a|}\,\rho^{|b-a-s|}\,\rho^{|s+t|} \\
&\qquad + 18\,\rho^{|s|}\,\rho^{2|b-a|}\,\rho^{|s+t|} \;+\; 18\,\rho^{2|b-a|}\,\rho^{|b-a+s+t|}\,\rho^{|b-a-s|} \\
&\qquad + 6\,\rho^{3|b-a|}\,\rho^{|b-a+t|}\,\big)\,.
\end{aligned} \tag{17}$$

Without further simplification we can directly obtain the various constituents of $\mathrm{var}(\hat{\rho}^\star)$ by the means of (17).

$$\begin{aligned}
\mathrm{var}(d) \;&=\; \mathrm{var}\left(\frac{1}{n}\sum_{i=1}^{n-1}\epsilon_i^3\epsilon_{i+1}\right) = (\text{put } s=1 \text{ and } t=0 \text{ in (17)}) \\
&=\; \frac{1}{n^2}\Bigg(9\sum_{a,b=1}^{n-1}\rho^{2|b-a|} \;+\; 9\sum_{a,b=1}^{n-1}\rho^{|b-a+1|}\,\rho^{|b-a-1|} \\
&\qquad + 18\,\rho\sum_{a,b=1}^{n-1}\rho^{|b-a|}\,\rho^{|b-a+1|} \;+\; 18\,\rho\sum_{a,b=1}^{n-1}\rho^{|b-a|}\,\rho^{|b-a-1|} \\
&\qquad + 18\,\rho^2\sum_{a,b=1}^{n-1}\rho^{2|b-a|} \;+\; 18\sum_{a,b=1}^{n-1}\rho^{2|b-a|}\,\rho^{|b-a+1|}\,\rho^{|b-a-1|} \\
&\qquad + 6\sum_{a,b=1}^{n-1}\rho^{4|b-a|}\Bigg)\,.
\end{aligned}$$

The first and the fifth sum is given by (7), the second sum by (8) and the third by (9). We find,

$$\sum_{a,b=1}^{n-1}\rho^{|b-a|}\,\rho^{|b-a-1|} = \sum_{a,b=1}^{n-1}\rho^{|b-a|}\,\rho^{|b-a+1|},$$

which is given by (9),

$$\sum_{a,b=1}^{n-1} \rho^{2|b-a|}\, \rho^{|b-a+1|}\, \rho^{|b-a-1|} = (n-1)\,\rho^2 \;+\; 2\sum_{i=1}^{n-2} i\,\rho^{4(n-i-1)}$$

$$= (n-1)\left(\frac{2}{1-\rho^4} + \rho^2 - 2\right) + 2\,\frac{(\rho^{4n-8} - \rho^{-4})}{(1-\rho^{-4})^2} \tag{18}$$

and

$$\begin{aligned}\sum_{a,b=1}^{n-1} \rho^{4|b-a|} &= (n-1) + 2\sum_{i=1}^{n-2} i\,\rho^{4(n-i-1)} \\ &= (n-1)\left(\frac{2}{1-\rho^4} - 1\right) + 2\,\frac{(\rho^{4n-8} - \rho^{-4})}{(1-\rho^{-4})^2}.\end{aligned} \tag{19}$$

(7), (8), (9), (18) and (19) give var(d). Further,

$$\begin{aligned}\mathrm{cov}(w,d) &= \mathrm{cov}\left(\frac{1}{n}\sum_{i=1}^{n-1}\epsilon_i^4,\, \frac{1}{n}\sum_{i=1}^{n-1}\epsilon_i^3\epsilon_{i+1}\right) \\ &= (\text{put } s=0 \text{ and } t=1 \text{ in } (17)) \\ &= \frac{1}{n^2}\left(36\sum_{a,b=1}^{n-1}\rho^{|b-a|}\,\rho^{|b-a+1|} \;+\; 36\,\rho\sum_{a,b=1}^{n-1}\rho^{2|b-a|}\right. \\ &\qquad \left. + 24\sum_{a,b=1}^{n-1}\rho^{3|b-a|}\,\rho^{|b-a+1|}\right).\end{aligned}$$

The last sum of this expression,

$$\begin{aligned}\sum_{a,b=1}^{n-1}\rho^{3|b-a|}\,\rho^{|b-a+1|} &= (n-1)\,\rho + \sum_{i=1}^{n-2} i\,\rho^{4(n-i)-3} \;+\; \sum_{i=1}^{n-2} i\,\rho^{4(n-i)-5} \\ &= (n-1)\left(\frac{1}{\rho^{-1}-\rho^3} \;+\; \frac{1}{\rho-\rho^5} \;-\; \frac{1}{\rho}\right) \\ &\quad + \frac{(\rho^{4n-7}+\rho^{4n-9})\,(1-\rho^{-4n+4})}{(1-\rho^{-4})^2}.\end{aligned} \tag{20}$$

(7), (9) and (20) give cov(w,d). Further,

$$\begin{aligned}\mathrm{var}(w) &= \mathrm{var}\left(\frac{1}{n}\sum_{i=1}^{n-1}\epsilon_i^4\right) = (\text{put } s=t=0 \text{ in } (17)) \\ &= \frac{1}{n^2}\left(72\sum_{a,b=1}^{n-1}\rho^{2|b-a|} \;+\; 24\sum_{a,b=1}^{n-1}\rho^{4|b-a|}\right),\end{aligned}$$

which is given from (7) and (19), respectively. We further need

$$\mathrm{E}(w) \quad = \quad \frac{3\,(n-1)}{n} \tag{21}$$

and

$$\mathrm{E}(d) \quad = \quad \frac{3\,\rho\,(n-1)}{n}. \tag{22}$$

As for the old estimator, the terms $\propto (n-1)^{-2}$ which contribute to $\mathrm{var}(\hat{\rho}^{\star})$ cancel out:

$$\mathrm{var}(\hat{\rho}^{\star}) \simeq \frac{5}{3}\frac{(1-\rho^2)}{(n-1)}. \tag{23}$$

2.2.2 Mean

We use (14) with d instead of c and w instead of v, respectively. We derive $\mathrm{cov}(w,d)$ from (7), (9) and (20), further, $\mathrm{var}(w)$ from (7) and (19), and we have $\mathrm{E}(w)$ and $\mathrm{E}(d)$ given by (21) and (22), respectively. This yields, up to the second order of deviations from the means of d and w,

$$\begin{aligned}\mathrm{E}(\hat{\rho}^{\star}) \quad \simeq \quad & \rho - \frac{1}{(n-1)}\,\frac{4}{3}\,\rho\left(5 - \frac{2}{1+\rho^2}\right) \\ & + \frac{1}{(n-1)^2}\left[4\,\frac{(\rho - \rho^{2n-1})}{(1-\rho^2)} + \frac{8}{3}\,\frac{(\rho^3 - \rho^{4n-1})}{(1-\rho^2)\,(1+\rho^2)^2}\right].\end{aligned} \tag{24}$$

2.3 Remark

$\hat{\rho}^{\star}$ is found to have a larger negative bias and also a larger variance than $\hat{\rho}$. In the simulation experiment, we intend not only to prove that. We are also interested to examine the significance of the term $\propto (n-1)^{-2}$ in (15). In the comparison we include the theoretical results of White (1961) who studied the old estimator (1) for process (3). He calculated the kth moment of (c/v) via the joint moment generating function $m(c,v)$, with c and v defined as in (4) without the factors $1/n$,

$$\mathrm{E}(\hat{\rho}^k) = \int_{-\infty}^{0}\int_{-\infty}^{v_k}\cdots\int_{-\infty}^{v_2} \left.\frac{\partial^k m(c,v)}{\partial c^k}\right|_{c=0} dv_1\, dv_2\, \ldots\, dv_k.$$

He expanded the integrand to terms of order n^{-3} and ρ^4 and gave the following results (the index 'W' refers to his study):

$$\mathrm{var}(\hat{\rho})_W \quad \simeq \quad \left(\frac{1}{n} - \frac{1}{n^2} + \frac{5}{n^3}\right) - \left(\frac{1}{n} - \frac{9}{n^2} + \frac{53}{n^3}\right)\rho^2 - \frac{12}{n^3}\,\rho^4, \tag{25}$$

$$\mathrm{E}(\hat{\rho})_W \quad \simeq \quad \left(1 - \frac{2}{n} + \frac{4}{n^2} - \frac{2}{n^3}\right)\rho + \frac{2}{n^2}\,\rho^3 + \frac{2}{n^2}\,\rho^5. \tag{26}$$

3 Simulation experiments

In the first experiment, for each combination of n and ρ listed in Table 1, we generated a set of 250 000 time series from process (3). For every time series, $\hat{\rho}^\star$ and $\hat{\rho}$ have been calculated. The sample means, $\mu_{\hat{\rho}^\star\,sim}$ and $\mu_{\hat{\rho}\,sim}$, and the sample standard deviations, $\sigma_{\hat{\rho}^\star\,sim}$ and $\sigma_{\hat{\rho}\,sim}$, over the simulations are listed in Table 1. Therein, our theoretical values—from (12), (15), (23) and (24)—are included. In case of the means, these are additionally splitted into terms $\propto (n-1)^{-2}$ and terms of lower order. Also the theoretical values from White's (1961) formula, herein (25) and (26), are listed.

We further investigated whether the distribution of $\hat{\rho}^\star$ approaches Gaussianity as fast as that of $\hat{\rho}$. Fig. 1 shows histograms of $\hat{\rho}^\star$, respectively $\hat{\rho}$, from the first simulation experiment in comparison with Gaussian distributions $N(\mu_{\hat{\rho}^\star\,sim}, \sigma^2_{\hat{\rho}^\star\,sim})$, respectively $N(\mu_{\hat{\rho}\,sim}, \sigma^2_{\hat{\rho}\,sim})$.

In the second simulation experiment (Fig. 2), formulas (12) versus (25) and (15) versus (26) are examined over the entire range of ρ, with particular interest on ρ large. We plot the negative bias, $\rho - \mathrm{E}(\hat{\rho})$, and $\sqrt{\mathrm{var}(\hat{\rho})}$. For each combination of ρ and n, the number of simulated time series after (3) is 250 000.

The error due to the limited number of simulations is small enough for not influencing any intended comparison.

4 Results and discussion

The first simulation experiment (Table 1) confirms, over a broad range of ρ and n, that $\hat{\rho}^\star$ has a larger bias and a larger variance, respectively, than $\hat{\rho}$.

In general, the deviations between the theoretical results and the simulation results decrease with n increased, as is to be expected.

For any combination of ρ and n listed in Table 1, our terms $\propto (n-1)^{-2}$ in (15), respectively (24), bring these theoretical results closer to the simulation results, with the exception of $\mathrm{E}(\hat{\rho})$ for $\rho = 0.9$ and $n = 800$ where the simulation noise prevents that comparison. For ρ large, respectively n small, these second order terms contribute heavier.

The frequency distribution of $\hat{\rho}^\star$ (Fig. 1) has a similar shape as that of $\hat{\rho}$. It is shifted to smaller values against that and also broader, reflecting the larger negative bias, respectively the larger variance. The functional form of the distribution of $\hat{\rho}$ is approximately the Leipnik distribution, which is heavier skewed for ρ large and tends to Gaussianity with n increased. Both estimators seem to approach Gaussianity equally fast (Fig. 1).

The second simulation experiment (Fig. 2) compares White's (1961) theoretical results for $\hat{\rho}$, (25) and (26), with ours, (12) and (15). It shows that his are better performing up to a certain value of ρ.

Above, our formulas describe better the simulation, particularly, the decline to zero negative bias, respectively zero variance, as ρ approaches unity. The decline to zero negative bias is caused by the term $\propto (n-1)^{-2}$ in (15). Those declines are reasonable, since for $\rho = 1$ all time series points have equal value and $c = v$ in (4).

For small ρ, his formulas perform better since they are more accurate with respect to powers of $(1/n)$ than ours. For larger ρ his approximation becomes less accurate. In particular, for $n \geq 3$ and $\rho > 0$, $\rho - \mathrm{E}(\hat{\rho})_W$ cannot become zero. For $n \geq 10$, $\frac{d}{d\rho}(\rho - \mathrm{E}(\hat{\rho})_W)$ cannot become negative. For $n \geq 8$, $\mathrm{var}(\hat{\rho})_W$ cannot become zero. That means, for those cases White's (1961) formulas cannot produce the decline to zero negative bias and zero variance, respectively.

The fact that Bartlett's (1946) formula for the variance, (13), describes the simulation better than (12) for $\rho \to 0$, is regarded as spurious.

These results mean that the expansion formulas for $\mathrm{var}(\hat{\rho})$ and $\mathrm{E}(\hat{\rho})$, (5) and (14), respectively, are sufficient to derive the principal behaviours for $\rho \to 1$. In case of the mean, however, no further approximation is allowed which would lead to the term $\propto (n-1)^{-2}$ in (15) be neglected.

5 Conclusions

1. In case of the stationary AR(1) process (3) with known mean, the new estimator, $\hat{\rho}^{\star}$, has a larger negative bias and a larger variance than the old estimator, $\hat{\rho}$. Its distribution tends to Gaussianity about equally fast.

2. Our formula (15) for $\mathrm{E}(\hat{\rho})$ describes the expected decline to zero negative bias for $\rho \to 1$.

3. The formula (12) for $\mathrm{var}(\hat{\rho})$ describes the expected decline to zero variance for $\rho \to 1$.

4. The second order Taylor expansion of a ratio is sufficient to derive these two principal behaviours for $\rho \to 1$, if no further approximations are made. This condition could be fulfilled since we had concentrated ourselves on the lag-1 estimator and process (3).

5. For unknown mean of the process, it is more complicated to derive the equations with the same accuracy.

Acknowledgements

Q. Yao is appreciated for comments on the manuscript. The present study was carried out while the author was Marie Curie Research Fellow (EU-TMR grant ERBFMBICT971919).

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Appendix

We assume that ϵ_i is drawn from a standard Gaussian distribution with serial correlations ρ_j. We derive (16) by means of the moment generating function. Write

$$\begin{aligned}\mathrm{cov}(\epsilon_a^3\epsilon_{a+s}, \epsilon_b^3\epsilon_{b+s+t}) &= \mathrm{E}(\epsilon_a^3\epsilon_{a+s}\epsilon_b^3\epsilon_{b+s+t}) - \mathrm{E}(\epsilon_a^3\epsilon_{a+s})\,\mathrm{E}(\epsilon_b^3\epsilon_{b+s+t}) \\ &= \mathrm{E}(\epsilon_a^3\epsilon_{a+s}\epsilon_b^3\epsilon_{b+s+t}) - 9\,\rho_s\,\rho_{s+t}.\end{aligned}$$

Now, $\mathrm{E}(\epsilon_a^3\epsilon_{a+s}\epsilon_b^3\epsilon_{b+s+t})$ is the coefficient of $(t_1^3\,t_2\,t_3^3\,t_4)/(3!\,1!\,3!\,1!)$ in the moment generating function

$$m(t_1, t_2, t_3, t_4) = \exp\left(\frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4}\sigma_{ij}t_it_j\right),$$

with

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{44} = 1,$$

$$\sigma_{12} = \sigma_{21} = \rho_s,$$

$$\sigma_{13} = \sigma_{31} = \rho_{b-a},$$

$$\sigma_{14} = \sigma_{41} = \rho_{b-a+s+t},$$

$$\sigma_{23} = \sigma_{32} = \rho_{b-a-s},$$

$$\sigma_{24} = \sigma_{42} = \rho_{b-a+t},$$

$$\sigma_{34} = \sigma_{43} = \rho_{s+t}.$$

Terms $\propto (t_1^3\, t_2\, t_3^3\, t_4)$ in $m(t_1, t_2, t_3, t_4)$ can only be within the fourth order of the series expansion of the exponential function, i. e., within

$$\frac{1}{4!}\left(\frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4}\sigma_{ij}t_it_j\right)^4,$$

further restricting, within

$$\begin{aligned}\frac{1}{384}(&t_1^2 \,+\, t_3^2 \,+\, 2\,\sigma_{12}t_1t_2 \,+\, 2\,\sigma_{13}t_1t_3\\ &+2\,\sigma_{14}t_1t_4 \,+\, 2\,\sigma_{23}t_2t_3 \,+\, 2\,\sigma_{24}t_2t_4 \,+\, 2\,\sigma_{34}t_3t_4)^4,\end{aligned}$$

further restricting, within

$$\begin{aligned}\frac{1}{384}(&4\,\sigma_{13}^2t_1^2t_3^2 \,+\, 2\,t_1^2t_3^2 \,+\, 4\,\sigma_{12}t_1^3t_2 \,+\, 4\,\sigma_{13}t_1^3t_3 \,+\, 4\,\sigma_{14}t_1^3t_4 \,+\, 4\,\sigma_{23}t_1^2t_2t_3\\ &+4\,\sigma_{24}t_1^2t_2t_4 \,+\, 4\,\sigma_{34}t_1^2t_3t_4 \,+\, 4\,\sigma_{12}t_1t_2t_3^2 \,+\, 4\,\sigma_{13}t_1t_3^3 \,+\, 4\,\sigma_{14}t_1t_3^2t_4\\ &+4\,\sigma_{23}t_2t_3^3 \,+\, 4\,\sigma_{24}t_2t_3^2t_4 \,+\, 4\,\sigma_{34}t_3^3t_4 \,+\, 8\,\sigma_{12}\sigma_{13}t_1^2t_2t_3 \,+\, 8\,\sigma_{12}\sigma_{14}t_1^2t_2t_4\\ &+8\,\sigma_{12}\sigma_{34}t_1t_2t_3t_4 \,+\, 8\,\sigma_{13}\sigma_{14}t_1^2t_3t_4 \,+\, 8\,\sigma_{13}\sigma_{23}t_1t_2t_3^2 \,+\, 8\,\sigma_{13}\sigma_{24}t_1t_2t_3t_4\\ &+8\,\sigma_{13}\sigma_{34}t_1t_3^2t_4 \,+\, 8\,\sigma_{14}\sigma_{23}t_1t_2t_3t_4 \,+\, 8\,\sigma_{23}\sigma_{34}t_2t_3^2t_4)^2.\end{aligned}$$

Finally, these terms are

$$\begin{aligned}\frac{1}{384}(&2\cdot 4\,\sigma_{13}^2t_1^2t_3^2\cdot 8\,\sigma_{12}\sigma_{34}t_1t_2t_3t_4 \,+\, 2\cdot 4\,\sigma_{13}^2t_1^2t_3^2\cdot 8\,\sigma_{13}\sigma_{24}t_1t_2t_3t_4\\ &+2\cdot 4\,\sigma_{13}^2t_1^2t_3^2\cdot 8\,\sigma_{14}\sigma_{23}t_1t_2t_3t_4 \,+\, 2\cdot 2\,t_1^2t_3^2\cdot 8\,\sigma_{12}\sigma_{34}t_1t_2t_3t_4\\ &+2\cdot 2\,t_1^2t_3^2\cdot 8\,\sigma_{13}\sigma_{24}t_1t_2t_3t_4 \,+\, 2\cdot 2\,t_1^2t_3^2\cdot 8\,\sigma_{14}\sigma_{23}t_1t_2t_3t_4\\ &+2\cdot 4\,\sigma_{12}t_1^3t_2\cdot 4\,\sigma_{34}t_3^3t_4 \,+\, 2\cdot 4\,\sigma_{13}t_1^3t_3\cdot 4\,\sigma_{24}t_2t_3^2t_4\\ &+2\cdot 4\,\sigma_{13}t_1^3t_3\cdot 8\,\sigma_{23}\sigma_{34}t_2t_3^2t_4 \,+\, 2\cdot 4\,\sigma_{14}t_1^3t_4\cdot 4\,\sigma_{23}t_2t_3^3\\ &+2\cdot 4\,\sigma_{23}t_1^2t_2t_3\cdot 4\,\sigma_{14}t_1t_3^2t_4 \,+\, 2\cdot 4\,\sigma_{23}t_1^2t_2t_3\cdot 8\,\sigma_{13}\sigma_{34}t_1t_3^2t_4\\ &+2\cdot 4\,\sigma_{24}t_1^2t_2t_4\cdot 4\,\sigma_{13}t_1t_3^3 \,+\, 2\cdot 4\,\sigma_{34}t_1^2t_3t_4\cdot 4\,\sigma_{12}t_1t_2t_3^2\\ &+2\cdot 4\,\sigma_{34}t_1^2t_3t_4\cdot 8\,\sigma_{13}\sigma_{23}t_1t_2t_3^2 \,+\, 2\cdot 4\,\sigma_{12}t_1t_2t_3^2\cdot 8\,\sigma_{13}\sigma_{14}t_1^2t_3t_4\\ &+2\cdot 4\,\sigma_{13}t_1t_3^3\cdot 8\,\sigma_{12}\sigma_{14}t_1^2t_2t_4 \,+\, 2\cdot 4\,\sigma_{14}t_1t_3^2t_4\cdot 8\,\sigma_{12}\sigma_{13}t_1^2t_2t_3\\ &+2\cdot 8\,\sigma_{12}\sigma_{13}t_1^2t_2t_3\cdot 8\,\sigma_{13}\sigma_{34}t_1t_3^2t_4 \,+\, 2\cdot 8\,\sigma_{13}\sigma_{14}t_1^2t_3t_4\cdot 8\,\sigma_{13}\sigma_{23}t_1t_2t_3^2).\end{aligned}$$

Thus, we find

$$\begin{aligned}
\mathrm{E}(\epsilon_a^3\epsilon_{a+s}\epsilon_b^3\epsilon_{b+s+t}) &= \frac{3!\,1!\,3!\,1!}{384}(96\,\sigma_{12}\sigma_{34} \;+\; 96\,\sigma_{13}\sigma_{24} \;+\; 96\,\sigma_{14}\sigma_{23} \\
&\quad + 192\,\sigma_{12}\sigma_{13}\sigma_{14} \;+\; 192\,\sigma_{13}\sigma_{23}\sigma_{34} \\
&\quad + 192\,\sigma_{12}\sigma_{13}^2\sigma_{34} \;+\; 192\,\sigma_{13}^2\sigma_{14}^2\sigma_{23} \\
&\quad + 64\,\sigma_{13}^3\sigma_{24}).
\end{aligned}$$

This gives the final result

$$\begin{aligned}
\mathrm{cov}(\epsilon_a^3\epsilon_{a+s}, \epsilon_b^3\epsilon_{b+s+t}) &= 9\,\rho_{b-a}\,\rho_{b-a+t} \;+\; 9\,\rho_{b-a+s+t}\,\rho_{b-a-s} \\
&\quad + 18\,\rho_s\,\rho_{b-a}\,\rho_{b-a+s+t} \;+\; 18\,\rho_{b-a}\,\rho_{b-a-s}\,\rho_{s+t} \\
&\quad + 18\,\rho_s\,\rho_{b-a}^2\,\rho_{s+t} \;+\; 18\,\rho_{b-a}^2\,\rho_{b-a+s+t}\,\rho_{b-a-s} \\
&\quad + 6\,\rho_{b-a}^3\,\rho_{b-a+t}.
\end{aligned}$$

It should be noted that this result has been checked using the joint cumulant of order eight, in my assessment without less effort.

Table 1: First simulation experiment. The number of simulated time series after (3) is in each case 250 000. *S.d.*: standard deviation. The order of the uncertainty of the mean, respectively the standard deviation, due to the limited number of simulations, is *S.d.* / $\sqrt{250\,000}$. Sim: simulation (sample mean and sample standard deviation, $\mu_{\hat{\rho}^\star\ sim}$ and $\sigma_{\hat{\rho}^\star\ sim}$, respectively $\mu_{\hat{\rho}\ sim}$ and $\sigma_{\hat{\rho}\ sim}$). T(24, 23): theoretical value, this study, (24), respectively (23). 2nd: term $\propto (n-1)^{-2}$ in (15), respectively (24). 1st: terms not $\propto (n-1)^{-2}$ in (15), respectively (24).

		$\hat{\rho}^\star$ Sim	$\hat{\rho}$ Sim	$\hat{\rho}^\star$ T(24) 1st	$\hat{\rho}^\star$ T(24) 2nd	$\hat{\rho}^\star$ T(24, 23)	$\hat{\rho}$ T(15) 1st	$\hat{\rho}$ T(15) 2nd	$\hat{\rho}$ T(15, 12)	$\hat{\rho}$ T(26, 25)
$\rho = .200$	*Mean*	.154 374 330	.168 676 974	.108 831 908	.010 541 716	.119 373 625	.155 555 555	.005 144 032	.160 699 588	.167 766 400
$n = 10$	*S.d.*	.322 500 703	.307 524 129			.421 637 021			.326 598 632	.304 073 675
	S.d. / 500	.000 645 001	.000 615 048							
$\rho = .200$	*Mean*	.171 464 244	.182 387 197	.156 815 114	.002 365 315	.159 180 430	.178 947 368	.001 154 201	.180 101 569	.181 991 600
$n = 20$	*S.d.*	.239 353 376	.216 955 360			.290 190 500			.224 780 594	.216 235 057
	S.d. / 500	.000 478 706	.000 433 910							
$\rho = .200$	*Mean*	.186 208 448	.192 241 088	.183 254 840	.000 355 634	.183 610 475	.191 836 734	.000 173 538	.192 010 273	.192 323 456
$n = 50$	*S.d.*	.162 273 411	.137 738 715			.180 701 580			.139 970 842	.137 720 319
	S.d. / 500	.000 324 546	.000 275 477							
$\rho = .500$	*Mean*	.388 294 770	.422 688 034	.248 148 148	.036 433 344	.284 581 493	.388 888 888	.016 460 842	.405 349 731	.422 125 000
$n = 10$	*S.d.*	.303 635 696	.291 799 352			.372 677 996			.288 675 134	.280 178 514
	S.d. / 500	.000 607 271	.000 583 598							
$\rho = .500$	*Mean*	.464 740 520	.480 871 032	.453 741 496	.001 229 117	.454 970 614	.479 591 836	.000 555 324	.480 147 160	.480 917 000
$n = 50$	*S.d.*	.143 100 872	.124 231 561			.159 719 141			.123 717 914	.124 209 500
	S.d. / 500	.000 286 201	.000 248 463							
$\rho = .500$	*Mean*	.486 428 042	.493 490 915	.484 787 472	.000 132 926	.484 920 398	.493 288 590	.000 060 057	.493 348 647	.493 435 814
$n = 150$	*S.d.*	.086 589 711	.071 039 083			.091 592 913			.070 947 565	.071 083 675
	S.d. / 500	.000 173 179	.000 142 078							

Table 1: (Continued)

		$\hat{\rho}^\star$ Sim	$\hat{\rho}$ Sim	$\hat{\rho}^\star$ T(24) 1st	$\hat{\rho}^\star$ T(24) 2nd	$\hat{\rho}^\star$ T(24, 23)	$\hat{\rho}$ T(15) 1st	$\hat{\rho}$ T(15) 2nd	$\hat{\rho}$ T(15, 12)	$\hat{\rho}$ T(26, 25)
$\rho = .900$	*Mean*	.794 824 765	.831 981 030	.653 998 255	.060 176 382	.714 174 637	.805 263 157	.025 764 012	.831 027 170	.825 372 450
$n = 20$	*S.d.*	.152 898 617	.141 649 998			.129 099 444			.100 000 000	.139 640 968
	S.d. / 500	.000 305 797	.000 283 299							
$\rho = .900$	*Mean*	.867 598 466	.883 239 327	.852 787 543	.002 251 858	.855 039 402	.881 818 181	.000 966 603	.882 784 785	.882 622 098
$n = 100$	*S.d.*	.056 144 769	.049 552 189			.056 556 637			.043 808 582	.049 831 684
	S.d. / 500	.000 112 289	.000 099 104							
$\rho = .900$	*Mean*	.894 646 055	.897 751 699	.894 150 146	.000 034 571	.894 184 717	.897 747 183	.000 014 839	.897 762 023	.897 759 744
$n = 800$	*S.d.*	.019 176 153	.015 732 535			.019 908 007			.015 420 675	.015 723 824
	S.d. / 500	.000 038 352	.000 031 465							
$\rho = .980$	*Mean*	.903 670 968	.929 367 351	.706 301 470	.182 799 711	.889 101 182	.876 842 105	.073 477 667	.950 319 772	.900 780 563
$n = 20$	*S.d.*	.117 565 805	.106 515 873			.058 937 969			.045 653 154	.118 185 438
	S.d. / 500	.000 235 131	.000 213 031							
$\rho = .980$	*Mean*	.964 341 773	.971 335 719	.953 867 979	.002 915 320	.956 783 299	.970 150 753	.001 249 438	.971 400 192	.970 390 010
$n = 200$	*S.d.*	.021 591 051	.019 022 430			.018 211 487			.014 106 557	.019 544 022
	S.d. / 500	.000 043 182	.000 038 044							
$\rho = .980$	*Mean*	.977 759 602	.979 033 841	.977 398 563	.000 028 899	.977 427 462	.979 019 509	.000 012 386	.979 031 895	.979 021 902
$n = 2000$	*S.d.*	.005 526 193	.004 652 735			.005 745 999			.004 450 831	.004 658 731
	S.d. / 500	.000 011 052	.000 009 305							
$\rho = .999$	*Mean*	.995 105 991	.996 604 567	.988 325 315	.006 226 665	.994 551 981	.994 995 991	.002 535 138	.997 531 130	.995 035 904
$n = 500$	*S.d.*	.005 699 764	.004 845 877			.002 583 928			.002 001 502	.005 953 760
	S.d. / 500	.000 011 399	.000 009 691							
$\rho = .999$	*Mean*	.997 587 240	.998 180 155	.996 335 334	.000 574 434	.996 909 768	.998 000 500	.000 245 543	.998 246 044	.998 002 994
$n = 2000$	*S.d.*	.001 870 816	.001 622 354			.001 290 994			.001 000 000	.001 728 444
	S.d. / 500	.000 003 741	.000 003 244							
$\rho = .999$	*Mean*	.998 907 925	.998 960 630	.998 893 464	.000 000 932	.998 894 397	.998 960 039	.000 000 399	.998 960 439	.998 960 043
$n = 50000$	*S.d.*	.000 247 577	.000 207 948			.000 258 136			.000 199 951	.000 207 779
	S.d. / 500	.000 000 495	.000 000 415							

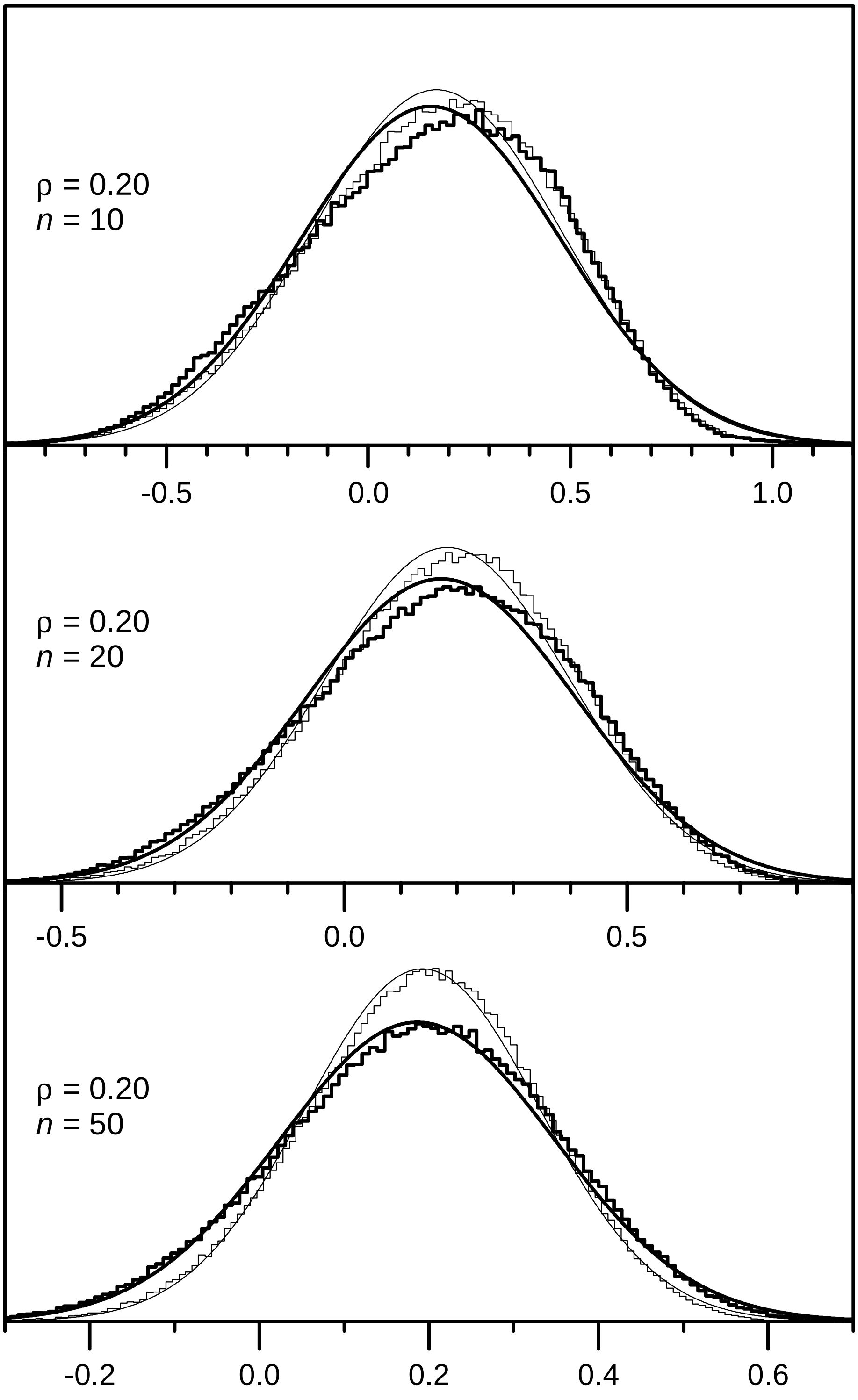

Figure 1: First simulation experiment (cf. Table 1). Histograms of $\hat{\rho}^{\star}$ (thick line), respectively $\hat{\rho}$ (thin line), compared with Gaussian distributions $N(\mu_{\hat{\rho}^{\star}\,sim}, \sigma^2_{\hat{\rho}^{\star}\,sim})$ (heavy line), respectively $N(\mu_{\hat{\rho}\,sim}, \sigma^2_{\hat{\rho}\,sim})$ (light line). The number of histogram classes follows Scott (1979).

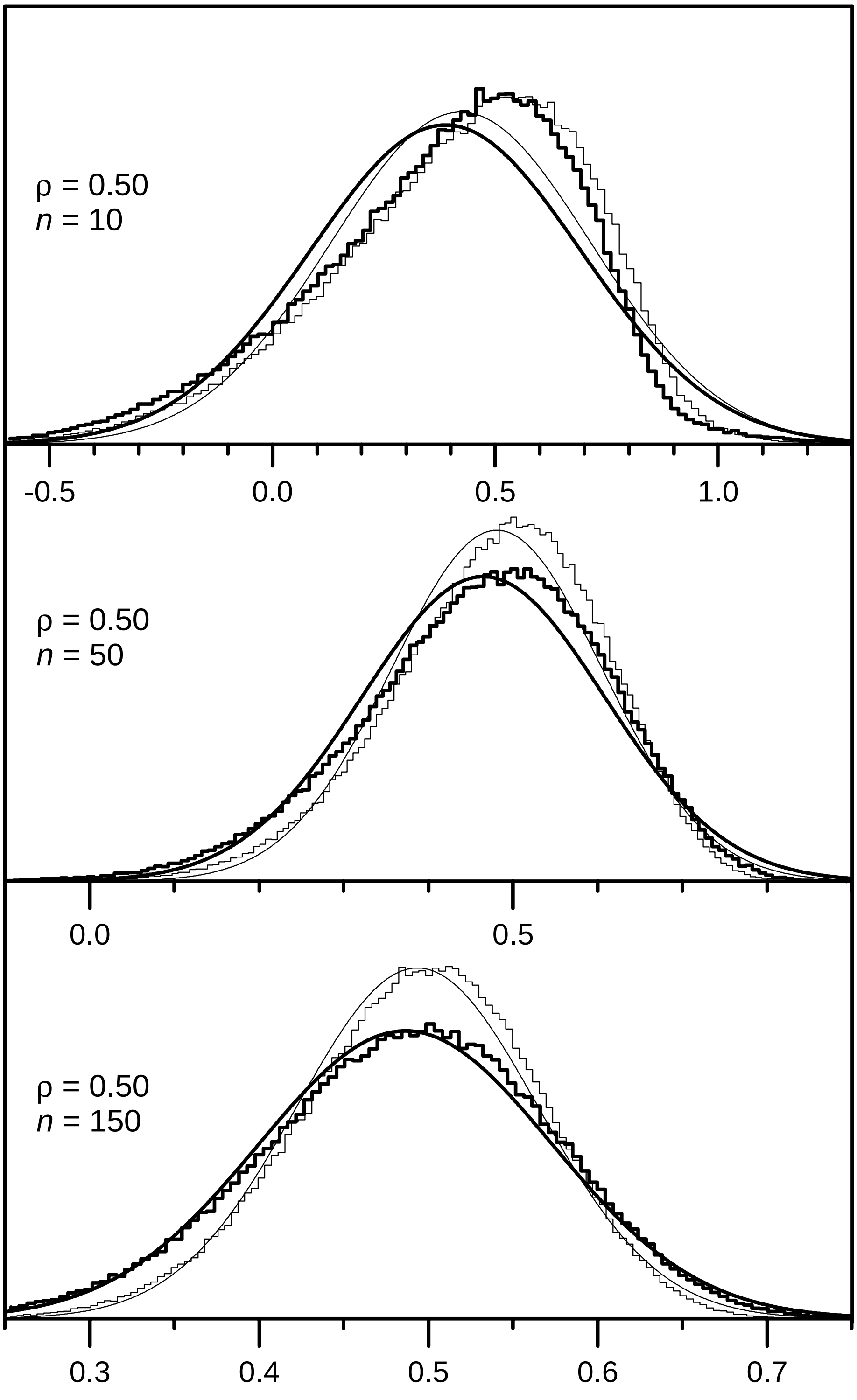

Figure 1: (Continued)

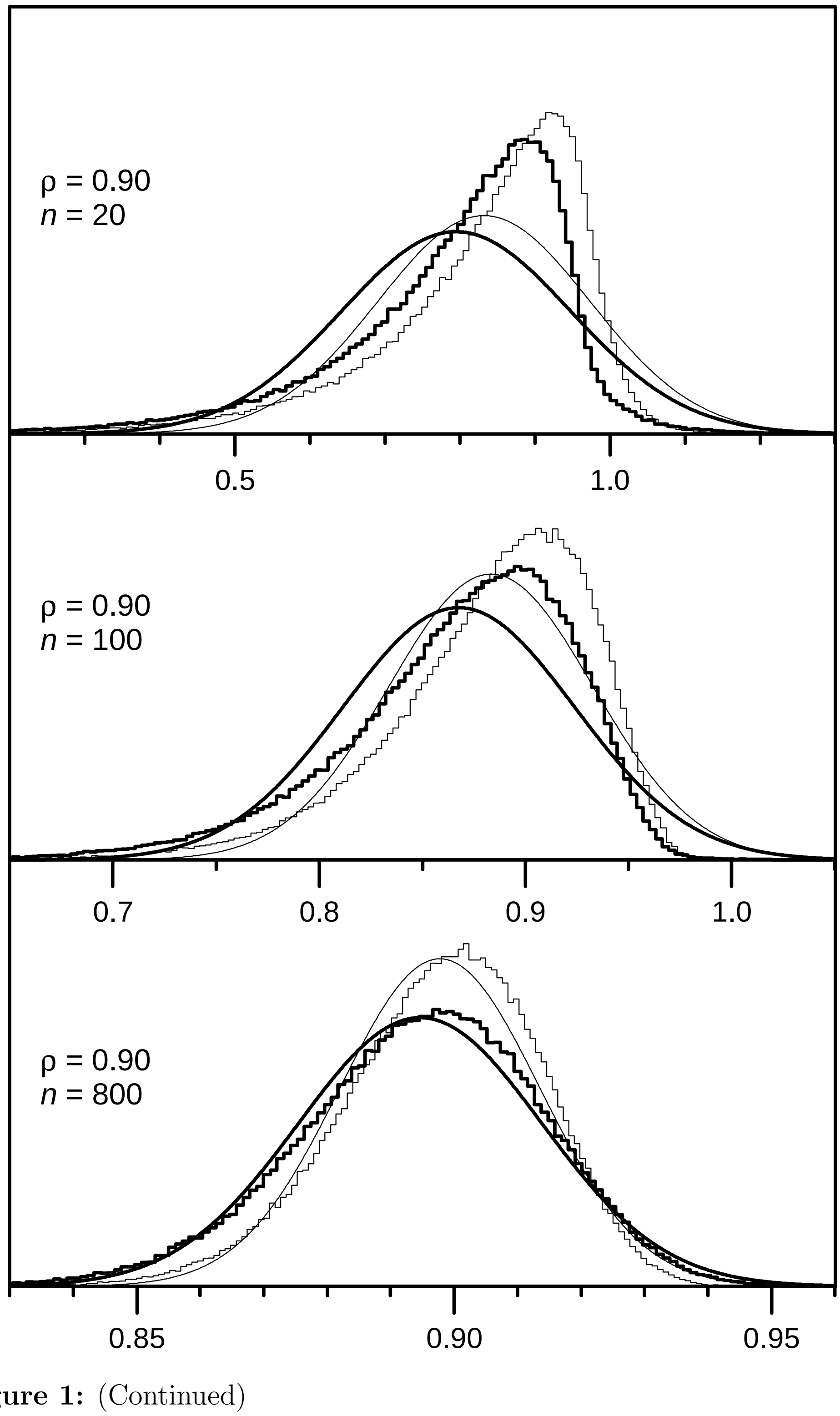

Figure 1: (Continued)

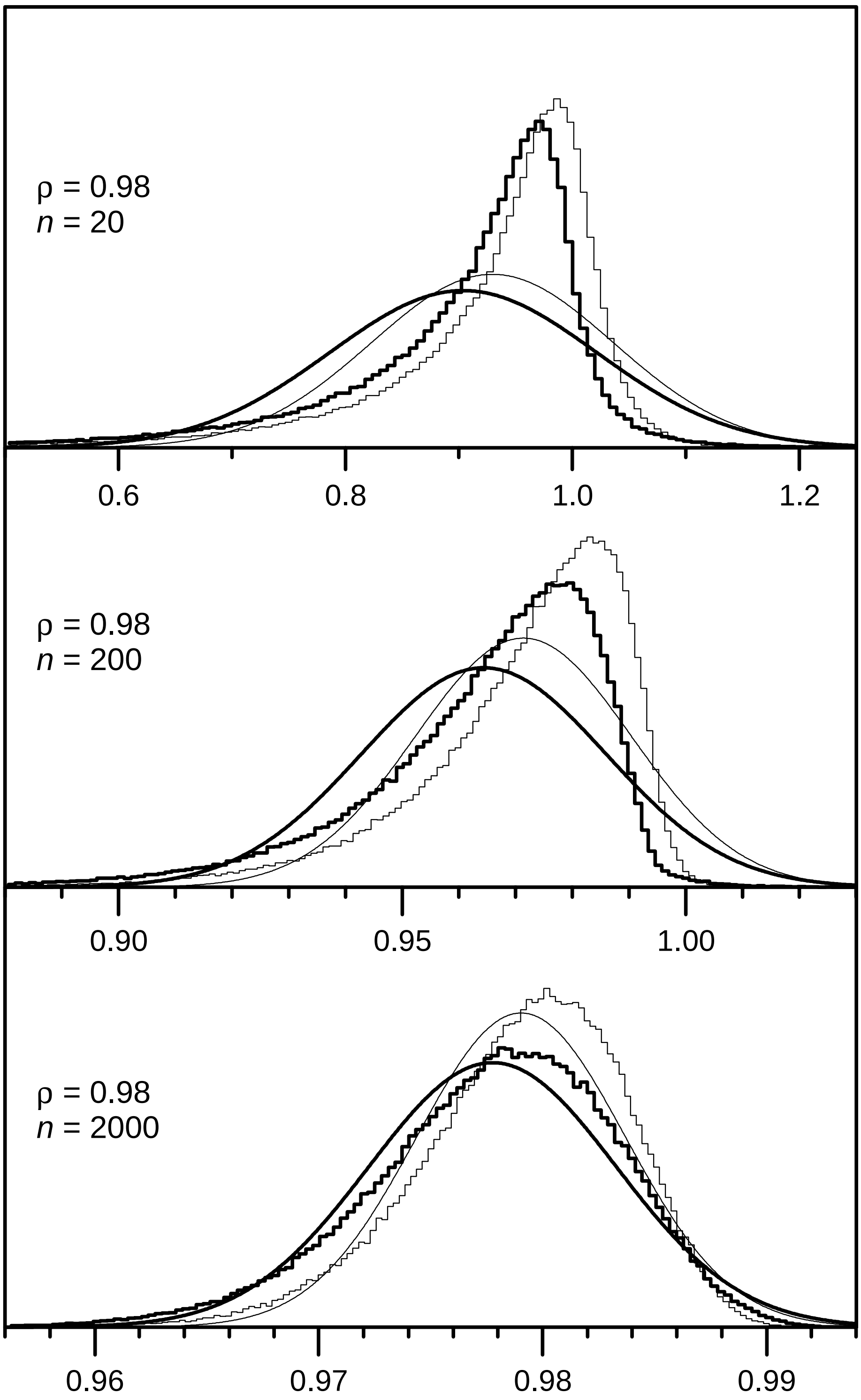

Figure 1: (Continued)

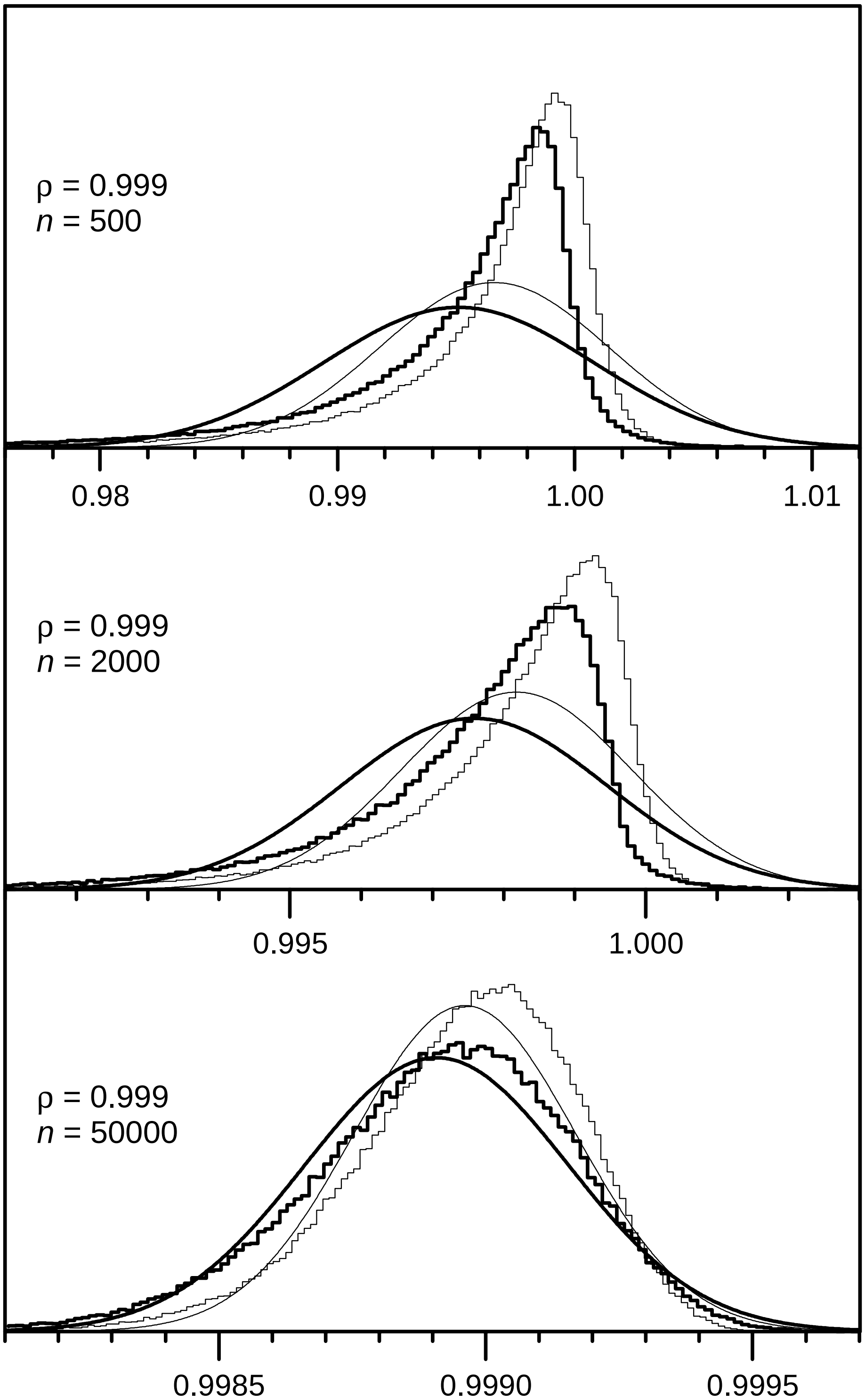

Figure 1: (Continued)

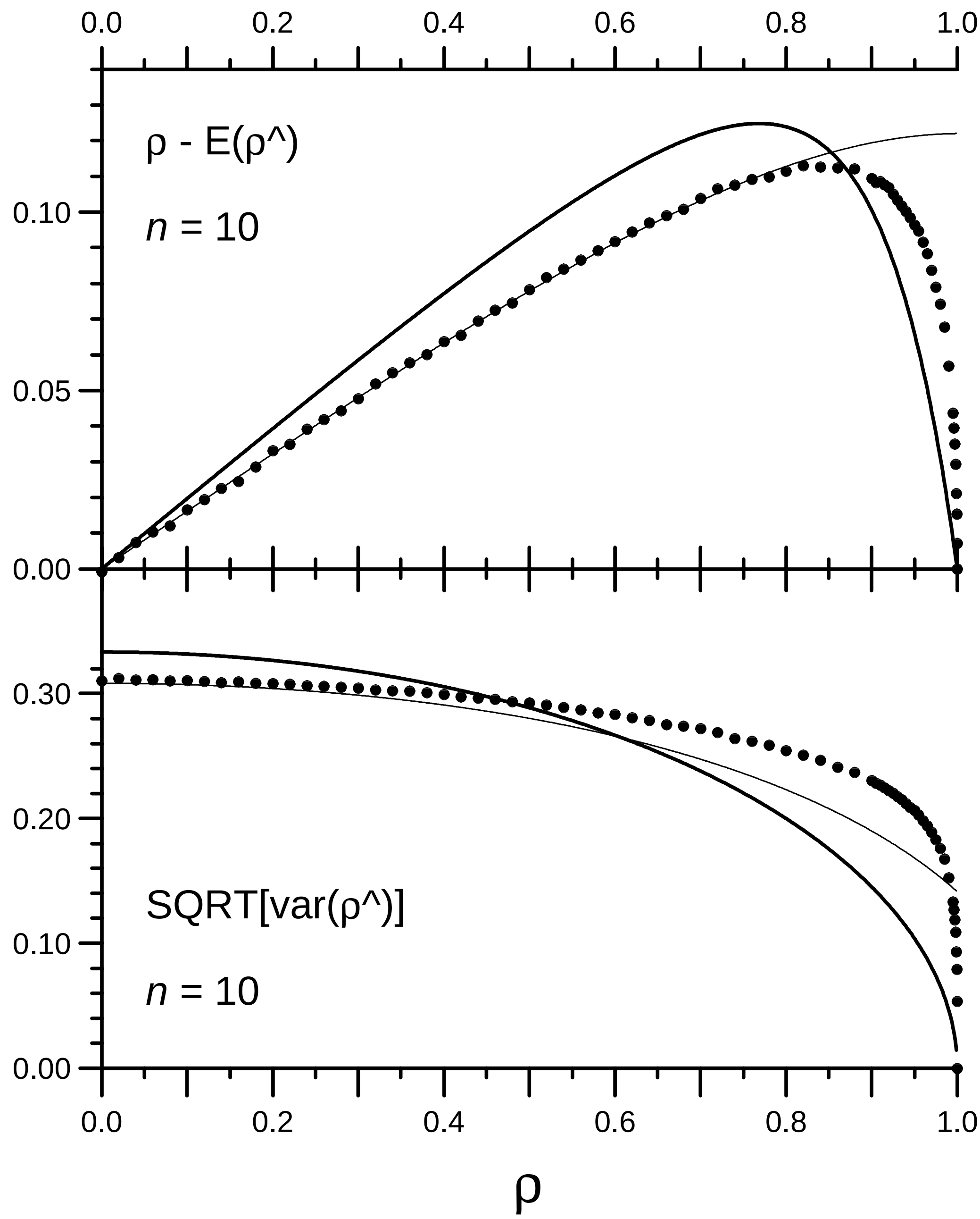

Figure 2: Second simulation experiment. Above: negative bias, below: standard deviation. Simulation results (dots). Theoretical result, our formula, (12) and (15), respectively, (heavy line). Theoretical result, White's (1961) formula, (25) and (26), respectively, (light line).

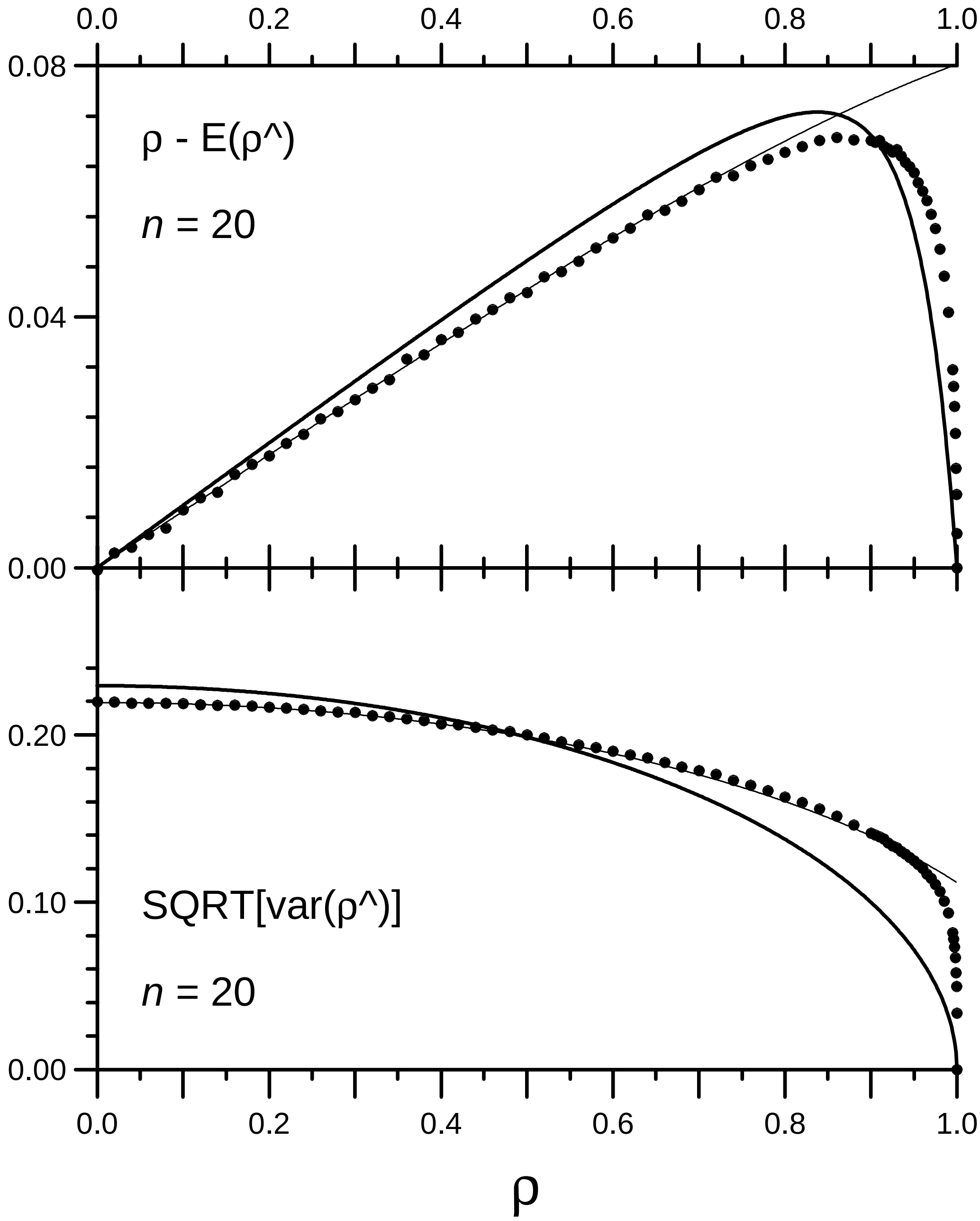

Figure 2: (Continued)

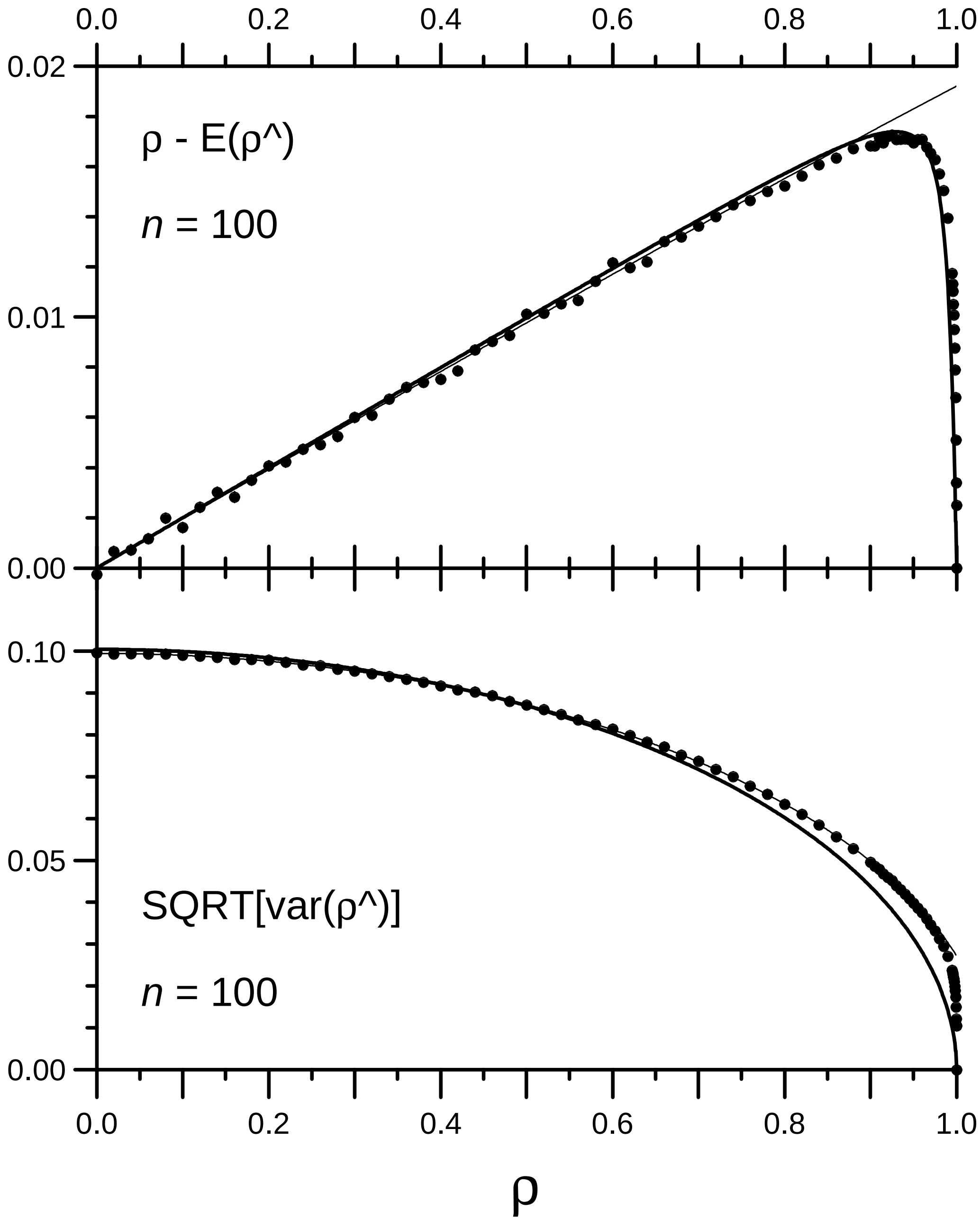

Figure 2: (Continued)