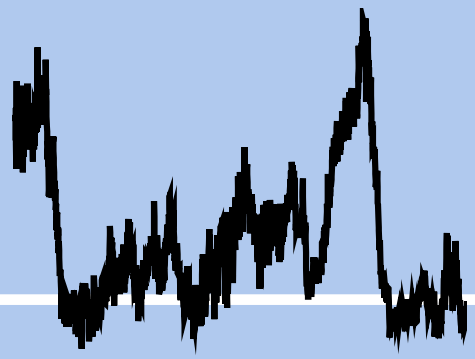
Climate Time Series Analysis:



Paleoclimate

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Grenoble, 30 January 2012

Prologue

Climate: knowledge and experience,
space–time scales

Time series analysis: statistical concept

Computer: implementation of concept

Mathematical formulas: no sweat!

Chapters

1. Introduction

1.1 Climate archives, variables and dating

1.2 Noise and statistical distribution

1.3 Persistence

1.4 Spacing

1.5 Aim and structure of this course

1. Introduction

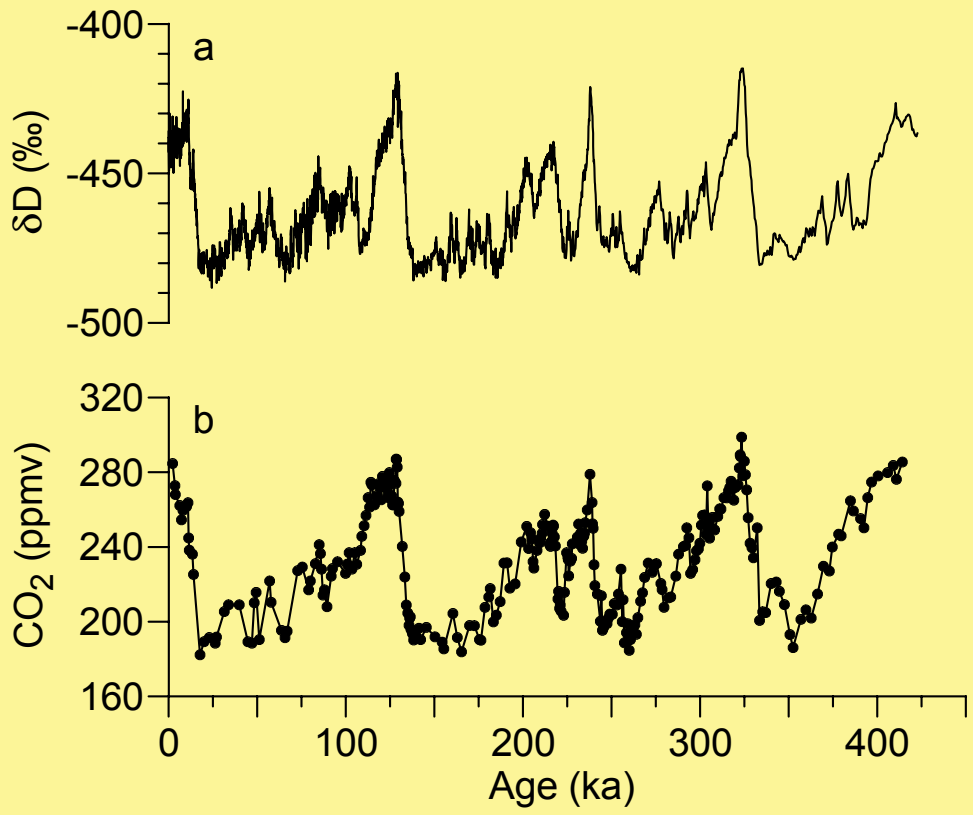


Figure 1.3. Ice core data: deuterium and CO_2 records from the Vostok station

$$\{t(i), x(i)\}_{i=1}^{n}$$

Time series sample

t, time value

x, climate-variable value

n, data size (sample size)

i, index

Climate time series

1. Introduction

$$X(T) = X_{\text{trend}}(T) + X_{\text{out}}(T) + S(T) \cdot X_{\text{noise}}(T), \qquad (1.1)$$

Climate equation, continuous time

T, time

X, climate variable observed

X_{trend}, trend component

X_{out}, outlier component

X_{noise}, noise component

S, variability

1. Introduction

$$X(i) = X_{\text{trend}}(i) + X_{\text{out}}(i) + S(i) \cdot X_{\text{noise}}(i), \qquad (1.2)$$

Climate equation, discrete time

$T(i)$, time points

$X(i) \equiv X(T(i))$, etc.

1. Introduction

Stochastic process $X(i)$ "Process level"

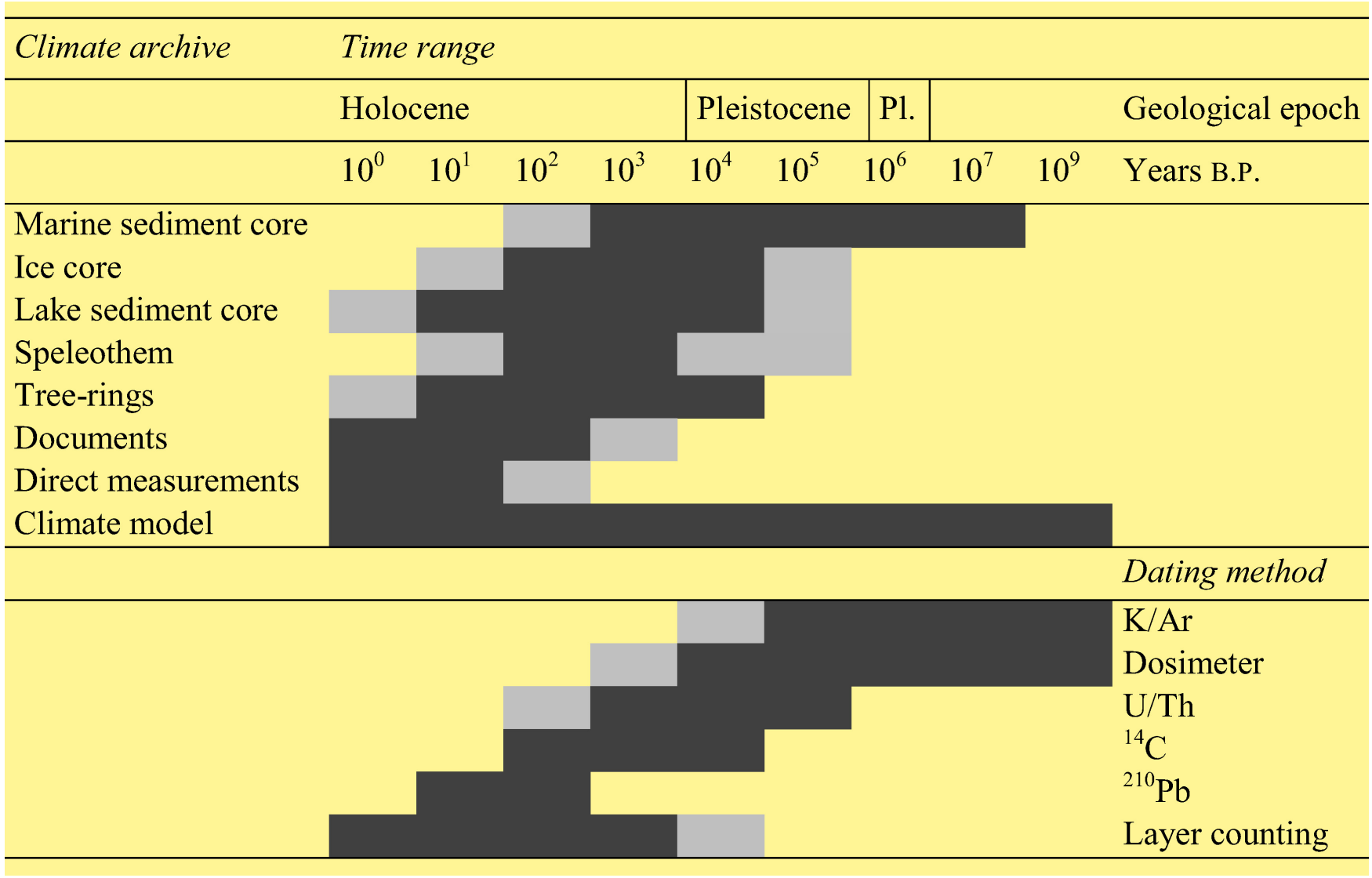
Time series $\{t(i), x(i)\}_{i=1}^{n}$ "Sample level"

Time series analysis uses sample to learn about process:
trend parameters, probability of extremes, cycles, etc.

Univariate: $T(i)$, $X(i)$
Bivariate: $T(i)$, $X(i)$, $Y(i)$

1. Introduction

1.1 Climate archives, variables and dating



Book, Table 1.1

1.1 Climate archives, variables and dating

Climate archive	*Location*	*Time range* (a)	*Proxy variable*	*Resolution* (a)	*Climate variable*
Marine sediment core	Eastern equatorial Pacific	10^6	$\delta^{18}O$, benthic foraminifera	10^3	Ice volume, bottom water temperature
Ice core	Antarctica	10^5	CO_2, air bubbles	10^3	CO_2, atmosphere
			δD, ice	10^2	Air temperature
	Greenland	10^5	SO_4 content, ice	10^0	Volcanic activity
			Ca content, ice	10^0	Aeolian dust, wind
			Dust content, ice	10^0	Aeolian dust, wind
			Conductivity,[a] ice	10^0	Soluble material, wind
			Na content, ice	10^0	Seasalt, wind
Tree-rings	Worldwide	10^4	$\Delta^{14}C$, wood	10^0	Solar irradiance, ocean circulation
Lake sediment core	Boston area	10^3	Varve thickness	10^0	Wind[b]
Speleothem	Southern Oman	10^4	$\delta^{18}O$, carbonate	10^1	Monsoon rainfall

1.1 Climate archives, variables and dating

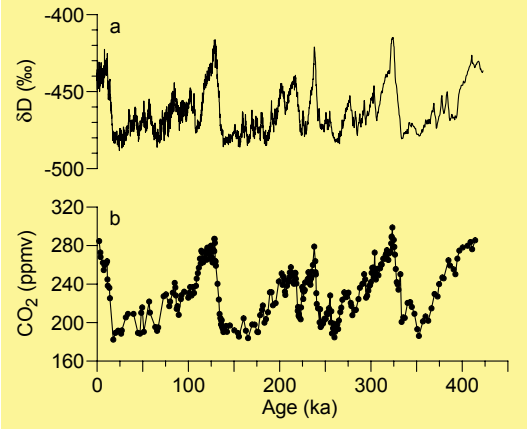


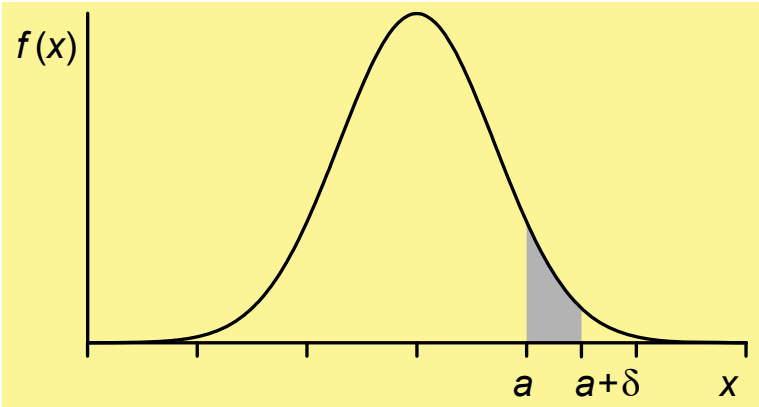
Figure 1.3. Ice core data: deuterium and CO_2 records from the Vostok station (Antarctica) over the past 420,000 years. The core was drilled into the ice (diameter: 12 cm, length: 3623 m) and recovered in segments. The deuterium record (**a**) was measured on ice material using a mass spectrometer. Values are given in delta notation: $\delta D = [(D/H)_{sample}/(D/H)_{SMOW} - 1] \cdot 1000‰$, where (D/H) is the number of D particles over the number of H particles and SMOW is "Standard Mean Ocean Water" standard. Total number of points, n, is 3311. The CO_2 record (**b**) was measured on air bubbles enclosed in the ice. Values are given as "parts per million by volume," n is 283. In **b,** values (*dots*) are connected by lines; in **a,** only lines are shown. The present-day CO_2 concentration ($\sim$ 389 ppmv) is not recorded in **b.** The construction of the timescale (named GT4) was achieved using a model of the ice accumulation and flow. Besides glaciological constraints, it further assumed that the points at 110 and 390 ka correspond to dated stages in the marine isotope record. Construction of the CO_2 timescale required additional modelling because in the ice core, air bubbles are younger in age than ice at the same depth. One climatological question associated with the data is whether variations in CO_2 (the values in air bubbles presenting the atmospheric value accurately) lead over or lag behind those of deuterium (which indicate temperature variations, low δD meaning low temperature). (Data from Petit

1.2 Noise and statistical distribution

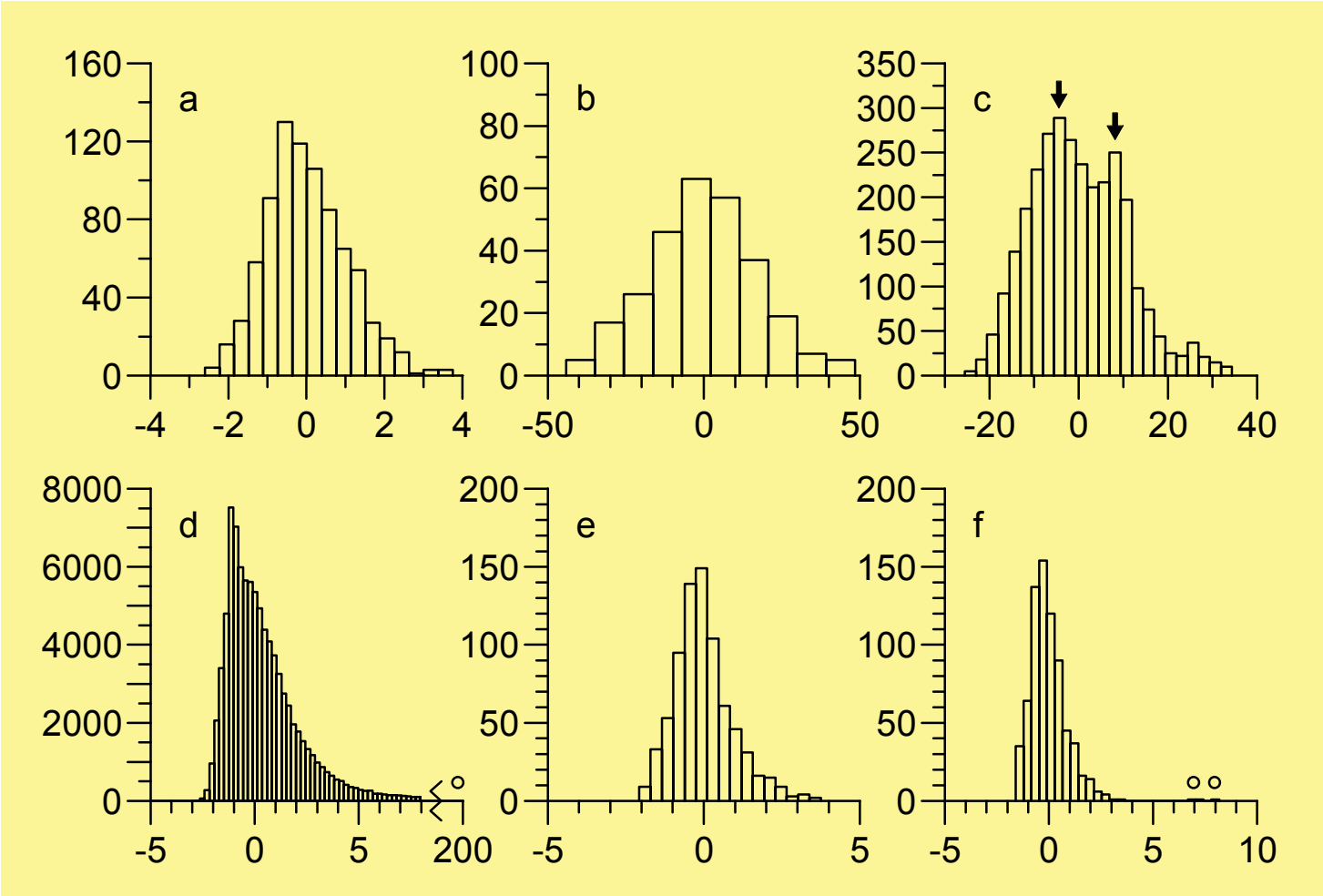
$$\text{prob}\left(a \leq X_{\text{noise}}(T) \leq a + \delta\right) \quad = \int_{a}^{a+\delta} f(x)dx, \qquad (1.3)$$

Probability density function (PDF), *f*(*x*)

Gaussian or normal PDF



1.2 Noise and statistical distribution



Histogram: inference of PDF

Residuals, "detrended version" of the $x(i)$

Book, Fig. 1.11 (**b,** Vostok CO_2; **c,** Vostok δD)

1.3 Persistence

$$E\left[X_{\text{noise}}(T_1) \cdot X_{\text{noise}}(T_2)\right] \text{ for } \quad T_1 \neq T_2$$

Autocovariance

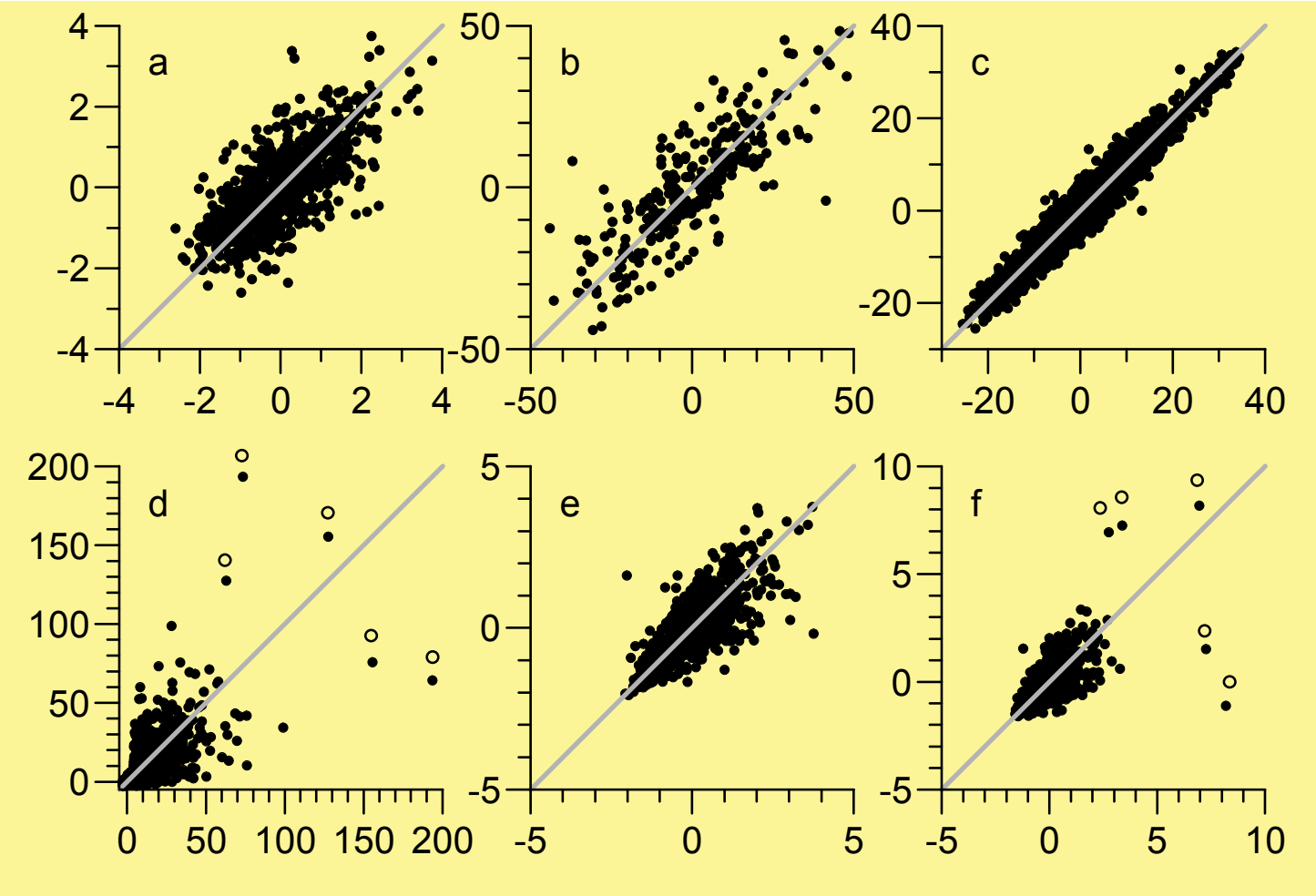
E, expectation operator (averaging)

$E[X_{\text{noise}}(T)] = 0$ (zero mean)

Positive autocovariance: persistence or “memory”

Book, p. 11–13

1.3 Persistence



Lag-1 scatterplot $x(i-1)$ vs $x(i)$: inference about persistence

Residuals, "detrended version" of the $x(i)$

Book, Fig. 1.12 (**b**, Vostok CO_2; **c**, Vostok δD)

1.3 Persistence

$$X(i) = X_{\text{trend}}(i) + X_{\text{out}}(i) + S(i) \cdot X_{\text{noise}}(i), \qquad (1.2)$$

$\{t(i), x(i)\}_{i=1}^{n}$

Residuals:
use sample to determine trend, outliers and variability,
then
subtract trend, remove outliers and divide by variability
(details: Chapters 3 and 4)

1.4 Spacing

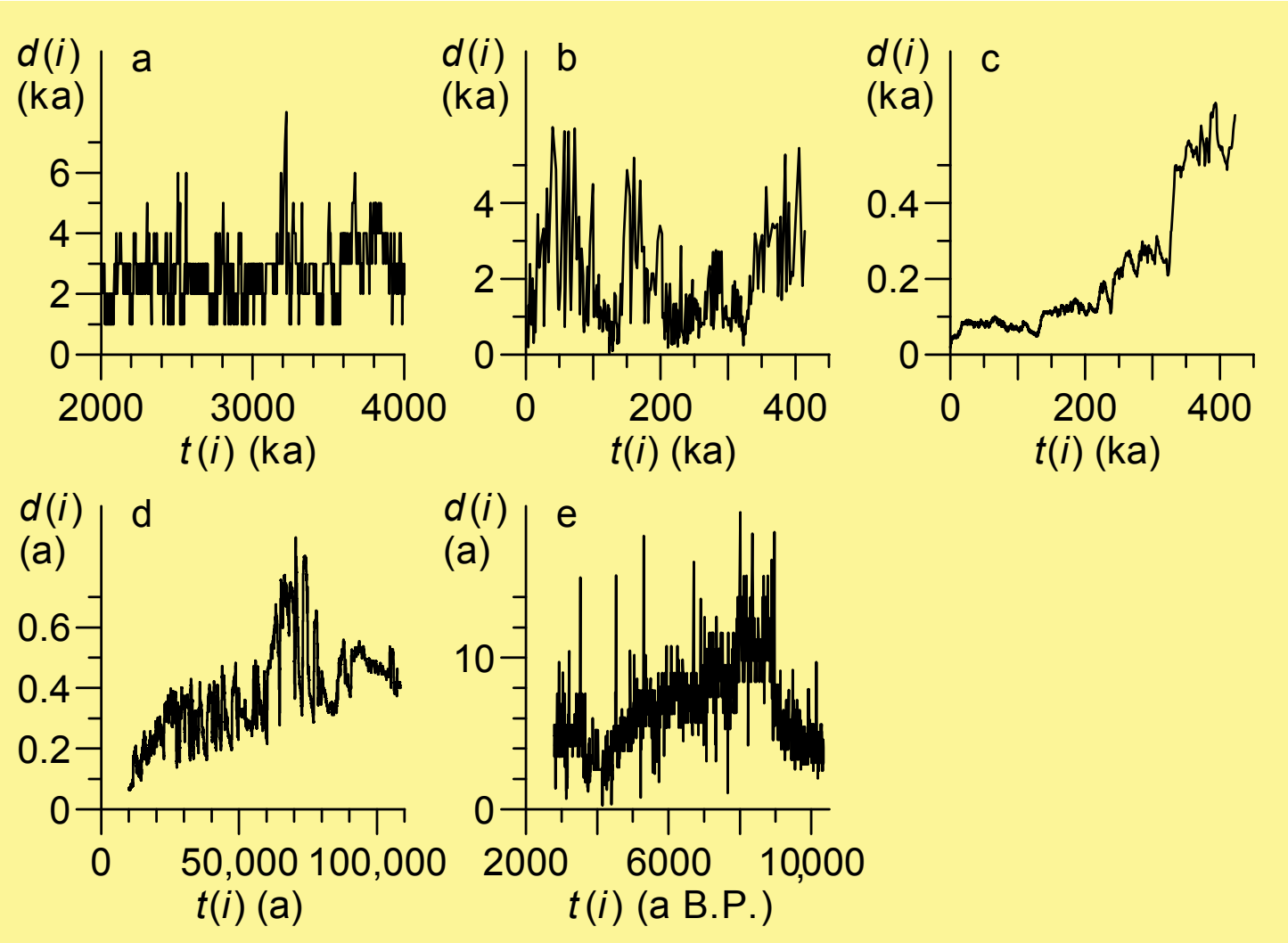
$$d(i) = t(i) - t(i-1)$$

Time spacing

$$\bar{d}$$

Average time spacing

1.4 Spacing



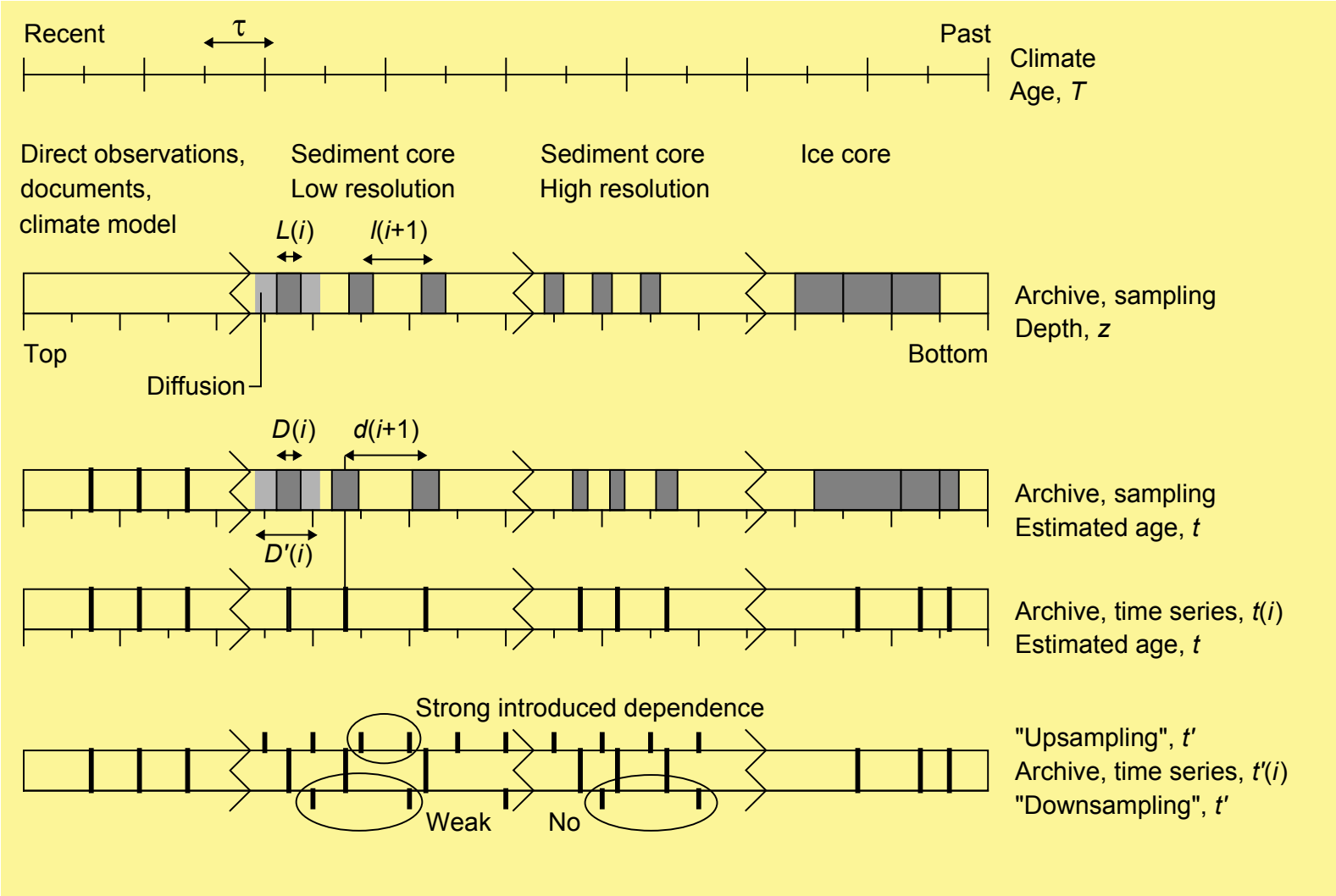
Book, Fig. 1.15 (**b,** Vostok CO_2; **c,** Vostok δD)

1.4 Spacing

$d(i) = d = \text{const.}$

Even spacing	Computer model output, observations
Uneven spacing	
Missing values	Observations, documentary data
Jittered	Observations, proxy data
Irregular	Proxy data

1.4 Spacing



1.5 Aim and structure of this course

$$X(i) = X_{\text{trend}}(i) + X_{\text{out}}(i) + S(i) \cdot X_{\text{noise}}(i), \qquad (1.2)$$

Climate equation, stochastic process $X(i)$

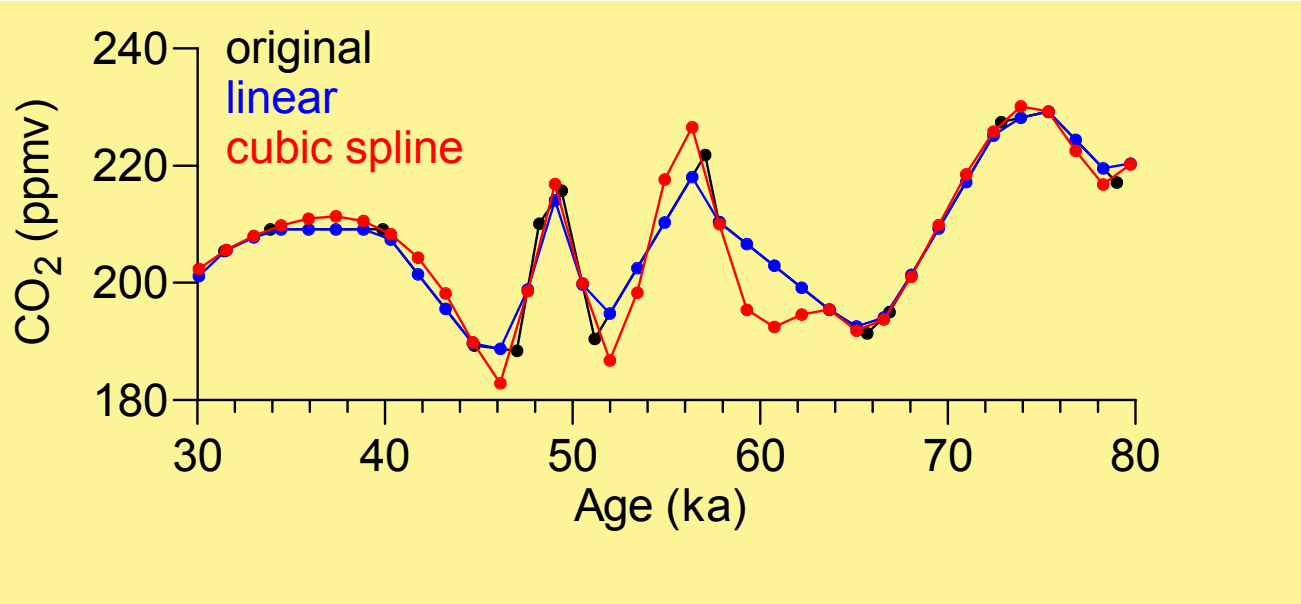
Time series $\{t(i), x(i)\}_{i=1}^{n}$

Time series analysis uses sample to learn about process: trend parameters, probability of extremes, cycles, etc.

Univariate: $T(i)$, $X(i)$

Bivariate: $T(i)$, $X(i)$, $Y(i)$

1.5 Aim and structure of this course



Interpolation: a step away from the original data

Book, Fig. 1.3 (modified)

2. Persistence Models

2.1 First-order autoregressive model

2.2

2.3

2.4

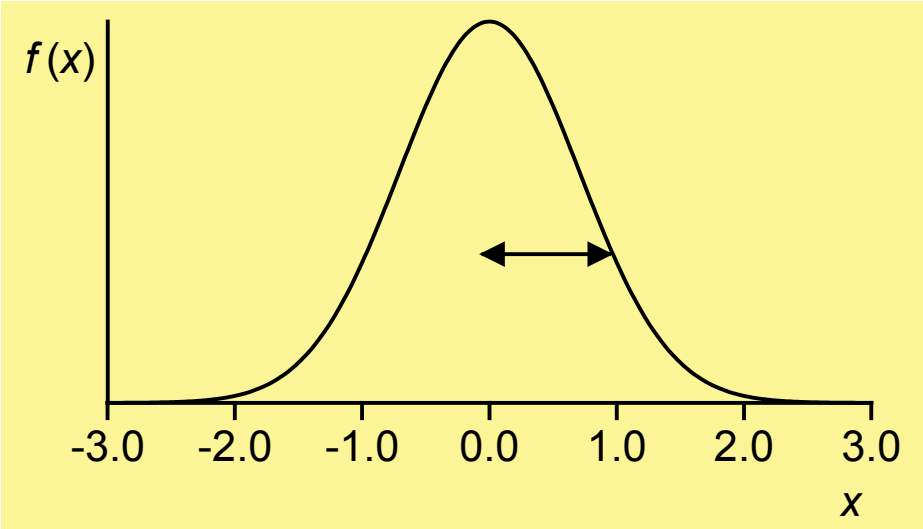
2.5

2.1 First-order autoregressive model

$$
\begin{aligned}
X_{\text{noise}}(1) &= \mathcal{E}_{\mathrm{N}(0,\,1)}(1), \\
X_{\text{noise}}(i) &= a \cdot X_{\text{noise}}(i-1) + \mathcal{E}_{\mathrm{N}(0,\,1-a^2)}(i), \qquad i = 2, \ldots, n.
\end{aligned}
\tag{2.1}
$$

AR(1) process, even spacing

$-1 < a < 1$



Innovation term $\mathcal{E}_{\mathrm{N}(\mu,\,\sigma^2)}(\cdot)$

Mean = μ = 0

Variance = σ^2 = 1

Gaussian shape

$f(x) = (2\pi)^{-1/2}\,\sigma^{-1}\exp[-(x-\mu)^2/(2\sigma^2)]$

$$f(x) = (2\pi)^{-1/2}\exp\left(-x^2/2\right)$$

2.1 First-order autoregressive model

$$
\begin{aligned}
X_{\text{noise}}(1) &= \mathcal{E}_{\mathrm{N}(0,\,1)}(1), \\
X_{\text{noise}}(i) &= a \cdot X_{\text{noise}}(i-1) + \mathcal{E}_{\mathrm{N}(0,\,1-a^2)}(i), \qquad i = 2, \ldots, n.
\end{aligned}
\tag{2.1}
$$

AR(1) process, even spacing

Mean: $E[X_{\text{noise}}(i)] = 0$ for all i

Variance: $VAR[X_{\text{noise}}(i)] = 1$ for all i

Strict stationarity

2.1 First-order autoregressive model

$$
\begin{aligned}
X_{\text{noise}}(1) &= \mathcal{E}_{\mathrm{N}(0,\,1)}(1), \\
X_{\text{noise}}(i) &= \exp\left\{-\left[T(i)-T(i-1)\right]/\tau\right\}\cdot X_{\text{noise}}(i-1) \\
&\quad + \mathcal{E}_{\mathrm{N}(0,\,1-\exp\{-2[T(i)-T(i-1)]/\tau\})}(i), \qquad i=2,\ldots,n.
\end{aligned}
\tag{2.9}
$$

AR(1) process, uneven spacing

Persistence time τ

- $\tau = 0$: no memory
- $\tau > 0$: memory

Heteroscedastic innovation term

If $T(i) - T(i-1) = d = \text{const.}$:

- even spacing with $a = \exp(-d/\tau)$

2.1 First-order autoregressive model

$$
\begin{aligned}
X_{\text{noise}}(1) &= \mathcal{E}_{\mathrm{N}(0,\,1)}(1), \\
X_{\text{noise}}(i) &= \exp\left\{-\left[T(i)-T(i-1)\right]/\tau\right\}\cdot X_{\text{noise}}(i-1) \\
&\quad + \mathcal{E}_{\mathrm{N}(0,\,1-\exp\{-2[T(i)-T(i-1)]/\tau\})}(i), \qquad i=2,\ldots,n.
\end{aligned}
\tag{2.9}
$$

AR(1) process, uneven spacing

Mean: $E[X_{\text{noise}}(i)] = 0$ for all i

Variance: $VAR[X_{\text{noise}}(i)] = 1$ for all i

Strict stationarity

2.1 First-order autoregressive model

$$E\left[X_{\text{noise}}(i+h)\cdot X_{\text{noise}}(i)\right]=\exp\left[-|T(i+h)-T(i)|/\tau\right].$$

Autocorrelation

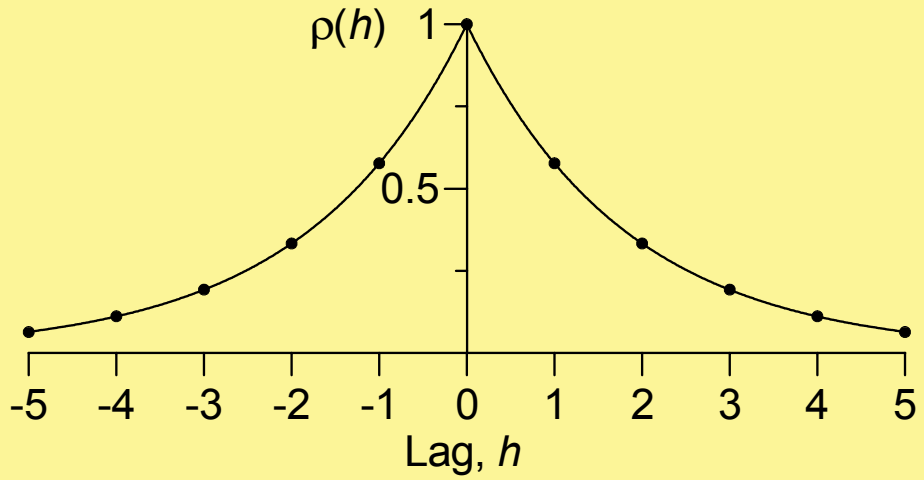


Figure 2.2. Autocorrelation function of the AR(1) process, $a > 0$. In the case of even

2.1 First-order autoregressive model

$$
\begin{aligned}
X_{\text{noise}}(1) &= \mathcal{E}_{\mathrm{N}(0,\,1)}(1), \\
X_{\text{noise}}(i) &= \exp\left\{-\left[T(i)-T(i-1)\right]/\tau\right\}\cdot X_{\text{noise}}(i-1) \\
&\quad + \mathcal{E}_{\mathrm{N}(0,\,1-\exp\{-2[T(i)-T(i-1)]/\tau\})}(i), \qquad i=2,\ldots,n.
\end{aligned}
\tag{2.9}
$$

AR(1) process, uneven spacing

$$
S(\widetilde{\tau}) = \sum_{i=2}^{n}\left[x_{\text{noise}}(i) - \exp\left\{-\left[t(i)-t(i-1)\right]/\widetilde{\tau}\right\}\cdot x_{\text{noise}}(i-1)\right]^2,
$$

Least-squares estimation

Residuals $x_{\text{noise}}(i)$, “detrended version” of the $x(i)$

Book, Eq. (2.11)

3. Bootstrap Confidence Intervals

3.1 Error bars and confidence intervals

3.2 Bootstrap principle

3.3 Bootstrap resampling

3.4 Bootstrap confidence intervals

3.5 Examples

3.6

3.1 Error bars and confidence intervals

$$X(i) = X_{\text{trend}}(i) + X_{\text{out}}(i) + S(i) \cdot X_{\text{noise}}(i), \qquad (1.2)$$

Climate equation, stochastic process *X*(*i*)

Time series $\{t(i), x(i)\}_{i=1}^{n}$

Time series analysis uses sample to learn about process: trend parameters, probability of extremes, cycles, etc.

3.1 Error bars and confidence intervals

Estimator

A recipe how to calculate parameter(s) of interest

Estimation

We apply recipe to data and get result.

Estimate

Numerical value of result

Hat notation

Denotes estimator/estimate.

3.1 Error bars and confidence intervals

True parameter	Sample	Estimator, estimate
	Time series	
θ	$\{t(i), x(i)\}_{i=1}^{n}$	$\hat{\theta}$
("Theta")		("Theta hat")
	$n < \infty$,	
	Observation errors (measurement, proxy, calibration, climate model, etc.)	$\hat{\theta} \neq \theta$

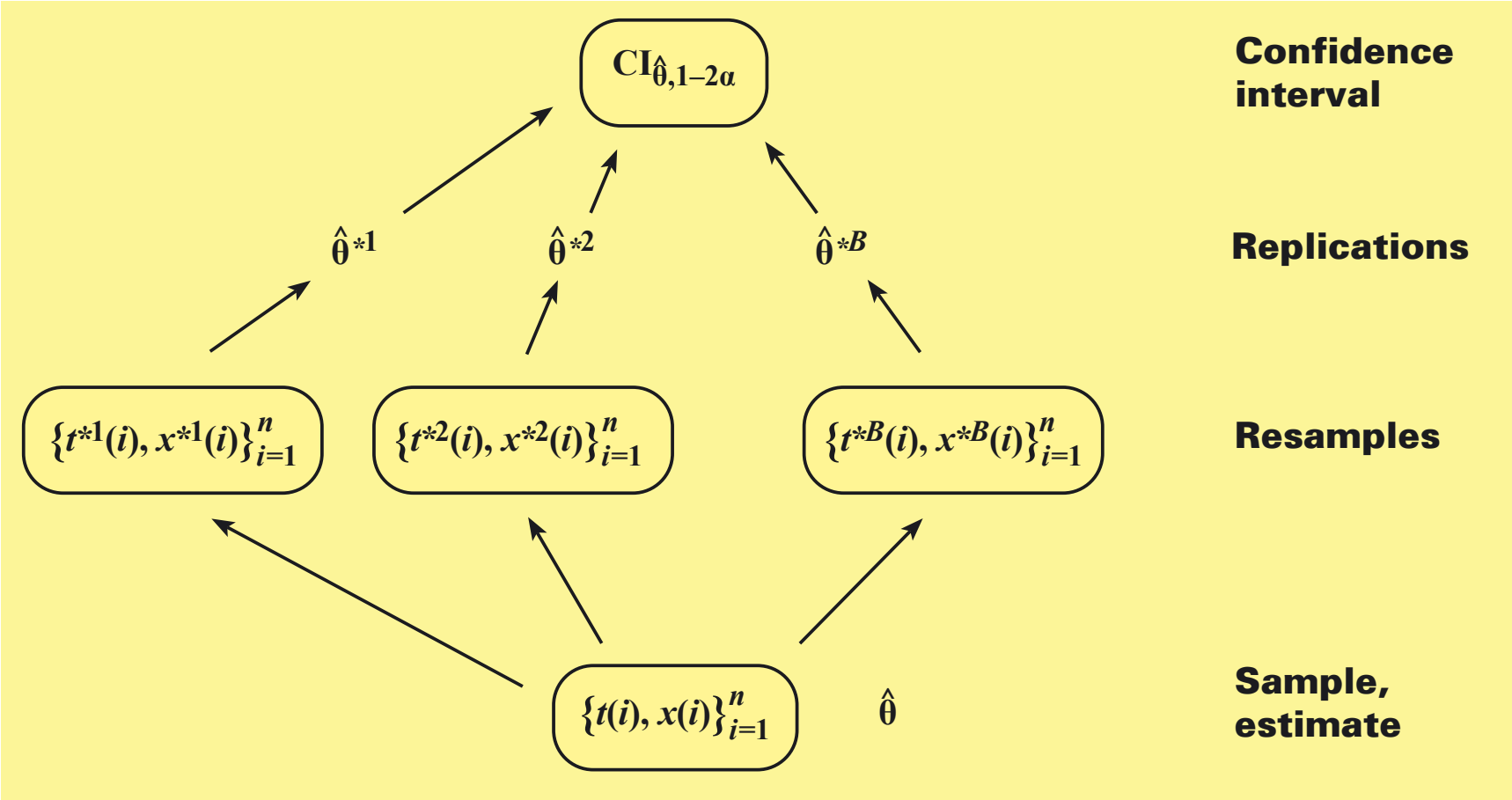
3.1 Error bars and confidence intervals

Estimate does not equal true (but unknown) value,
it deviates.

However,
we can say how large this deviation typically is:
error bars, confidence intervals, etc.

We need statistical language,
we make a statistical inference.

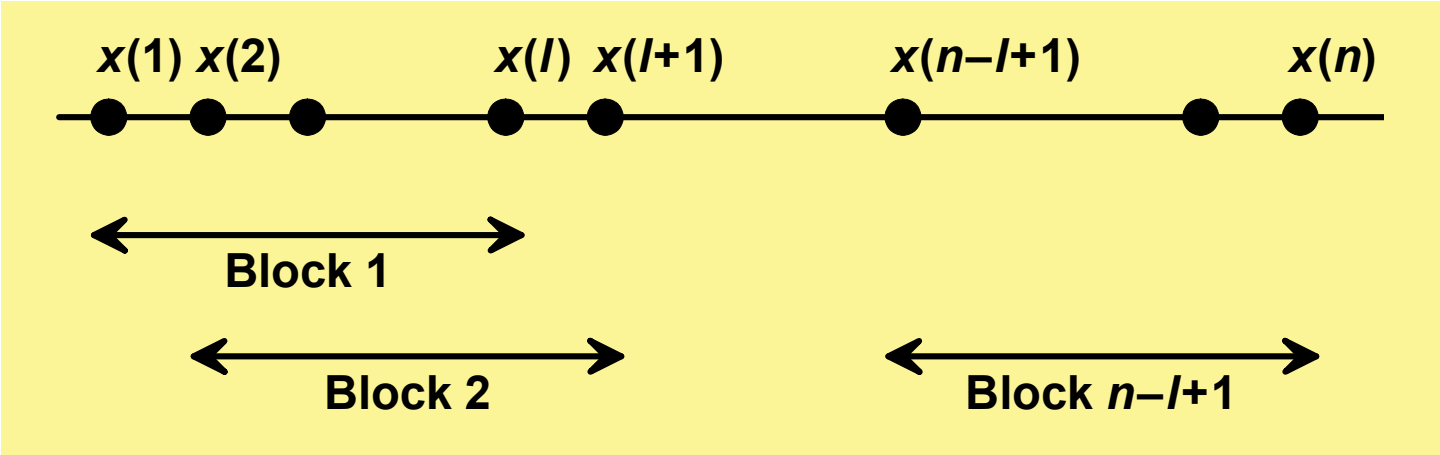
3.2 Bootstrap principle



3.2 Bootstrap principle

- Bootstrap
  - Resampling
    - Moving block bootstrap (MBB)
    - Autoregressive bootstrap (ARB)
    - Other
  - Confidence interval construction
    - Normal
    - Student's *t*
    - Percentile
    - BCa

3.3 Bootstrap resampling



Moving block bootstrap (MBB)

3.3 Bootstrap resampling

Step 1	Data	$\{t(i), x(i)\}_{i=1}^{n}$
Step 2	Resampled times	$\{t^*(i)\}_{i=1}^{n} = \{t(i)\}_{i=1}^{n}$
	unchanged	
Step 3	Residuals (Eq. 3.29)	$r(i) = [x(i) - \widehat{x}_{\text{trend}}(i) - \widehat{x}_{\text{out}}(i)] \Big/ \widehat{S}(i)$
Step 4	Apply MBB	
	(Algorithm 3.1)	
	to residuals	$\{r(i)\}_{i=1}^{n}$
Step 5	Resampled residuals	$\{r^*(i)\}_{i=1}^{n}$
Step 6	Use resampled residuals	
	to produce resamples	$x^*(i) = \widehat{x}_{\text{trend}}(i) + \widehat{x}_{\text{out}}(i) + \widehat{S}(i) \cdot r^*(i)$

Algorithm 3.3. MBB for realistic climate processes, which comprise trend, outlier and variability components.

3.4 Bootstrap confidence intervals

$$\widehat{\mathrm{se}}_{\widehat{\theta}^*} = \left\{ \sum_{b=1}^{B} \left[\widehat{\theta}^{*b} - \left\langle \widehat{\theta}^{*b} \right\rangle \right]^2 \Big/ (B-1) \right\}^{1/2}, \qquad (3.30)$$

Bootstrap standard error

3.4 Bootstrap confidence intervals

$$\mathrm{CI}_{\widehat{\theta},1-2\alpha} = \left[\widehat{\theta} + t_\nu(\alpha) \cdot \widehat{\mathrm{se}}_{\widehat{\theta}*}; \widehat{\theta} - t_\nu(\alpha) \cdot \widehat{\mathrm{se}}_{\widehat{\theta}*}\right], \qquad (3.32)$$

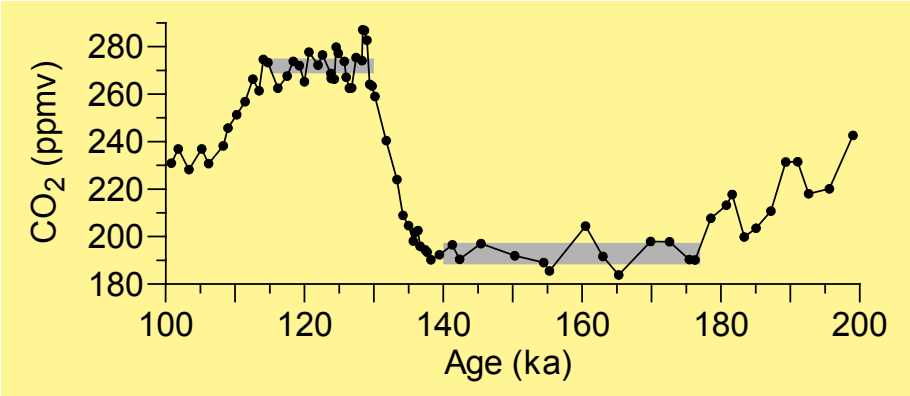
Student's *t* confidence interval

$t_v(\alpha)$ Percentage point of Student's *t*-distribution
with v degrees of freedom ($v = n$ – number of parameters)

Takes into account that not only standard error,
but also fit value estimated.

Negligible difference to normal CI if v above ~ 30

3.5 Examples



Vostok, CO_2

- Glacial, [140 ka; 177 ka], n = 13
- Interglacial, [115 ka; 130 ka], n = 24

Bootstrap resampling

- Block bootstrap (not MBB, but SB)
- Block length (average) $NINT\left(4 \cdot \tau / \bar{d}\right)$
- B = 2000

Confidence interval

- 95% Student's t

Book, Fig. 3.3; SB, stationary bootstrap (Politis and Romano 1994)

4. Regression I

4.1

4.2

4.3 Nonparametric regression or smoothing

4. Regression I

$$X(i) = X_{\text{trend}}(i) + X_{\text{out}}(i) + S(i) \cdot X_{\text{noise}}(i), \qquad (1.2)$$

Climate equation, stochastic process $X(i)$

Time series $\{t(i), x(i)\}_{i=1}^{n}$

Time series analysis uses sample to learn about process:
trend parameters, probability of extremes, cycles, etc.

Ignore $X_{\text{out}}(i)$,
focus on $X_{\text{trend}}(i)$ and also $S(i)$.

Brückner (1890), Hann (1901), Köppen (1923)

4. Regression I

$X_{\text{trend}}(i)$

Parametric trend model

- Linear
- Nonlinear

Nonparametric trend model

4.3 Nonparametric regression or smoothing

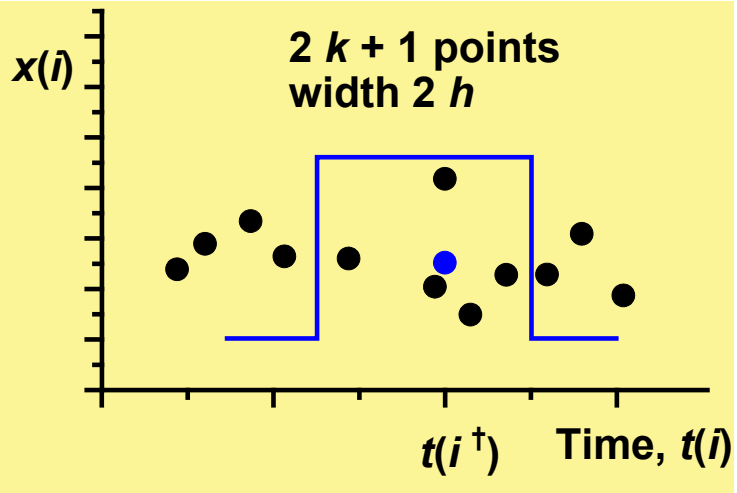
$X_{\text{trend}}(T)$ estimated via
averaging points x(i)
in neighbourhood around T

Short-term noise components smoothed away

No parametric trend model

4.3 Nonparametric regression or smoothing

Running mean



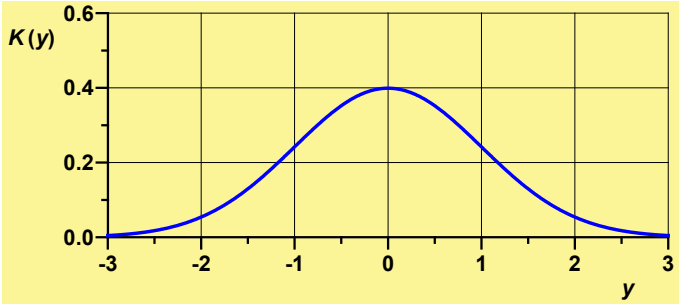
$$\widehat{X}^{\mathrm{PC}}_{\mathrm{trend}}(T) = h^{-1} \sum_{i=1}^{n} [T(i) - T(i-1)] \; K\left[\frac{T - T(i)}{h}\right] \; X(i), \qquad (4.52)$$

Kernel smoothing

K, kernel function (: *K* = 0 outside, *K* = const. > 0 inside)

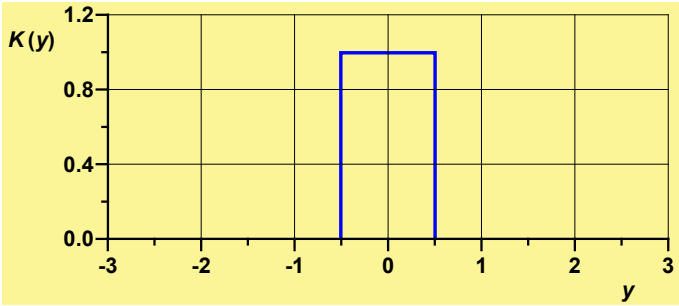
h, bandwidth

4.3 Nonparametric regression or smoothing



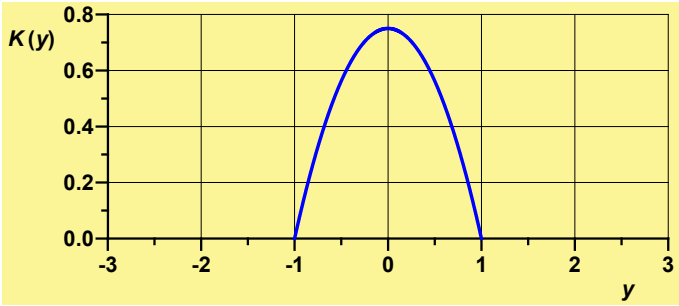
$K(y) = (2\pi)^{-1/2} \exp(-y^2/2)$

Gaussian kernel



$K(y) = 1$ for $|y| \leq 1/2$

Uniform kernel



$K(y) = 0.75\,(1 - y^2)$ for $|y| \leq 1$

Epanechnikov kernel

4.3 Nonparametric regression or smoothing

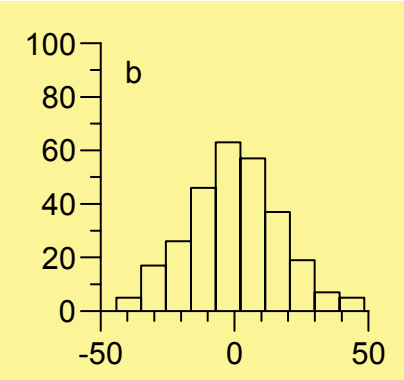
Smoothing problem

selection of bandwidth

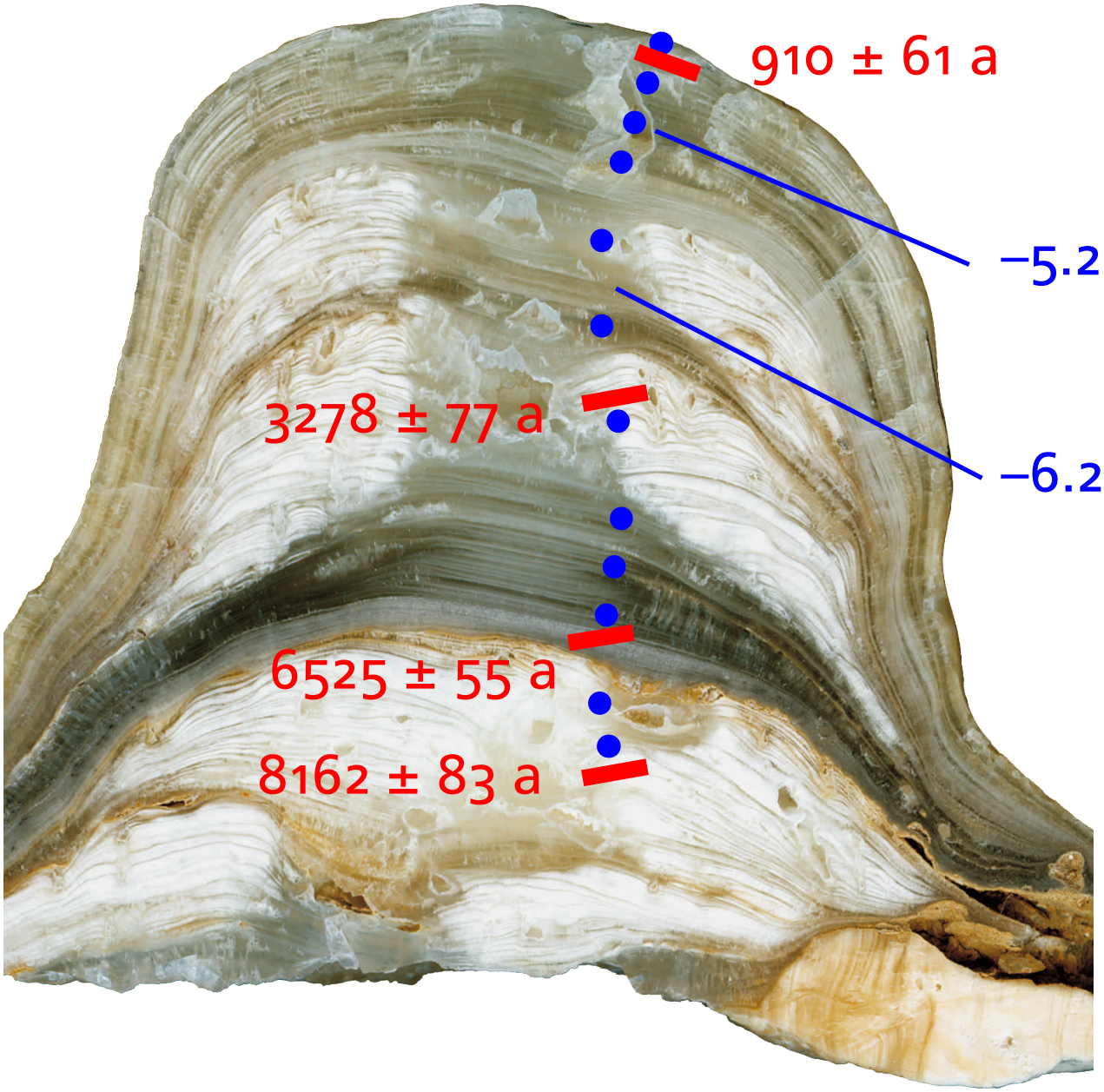
Bandwidth	Standard error	Bias
Large	Small (many points)	Large (loose details)
Small	Large	Small

(choice of *K* more of "cosmetic" interest)

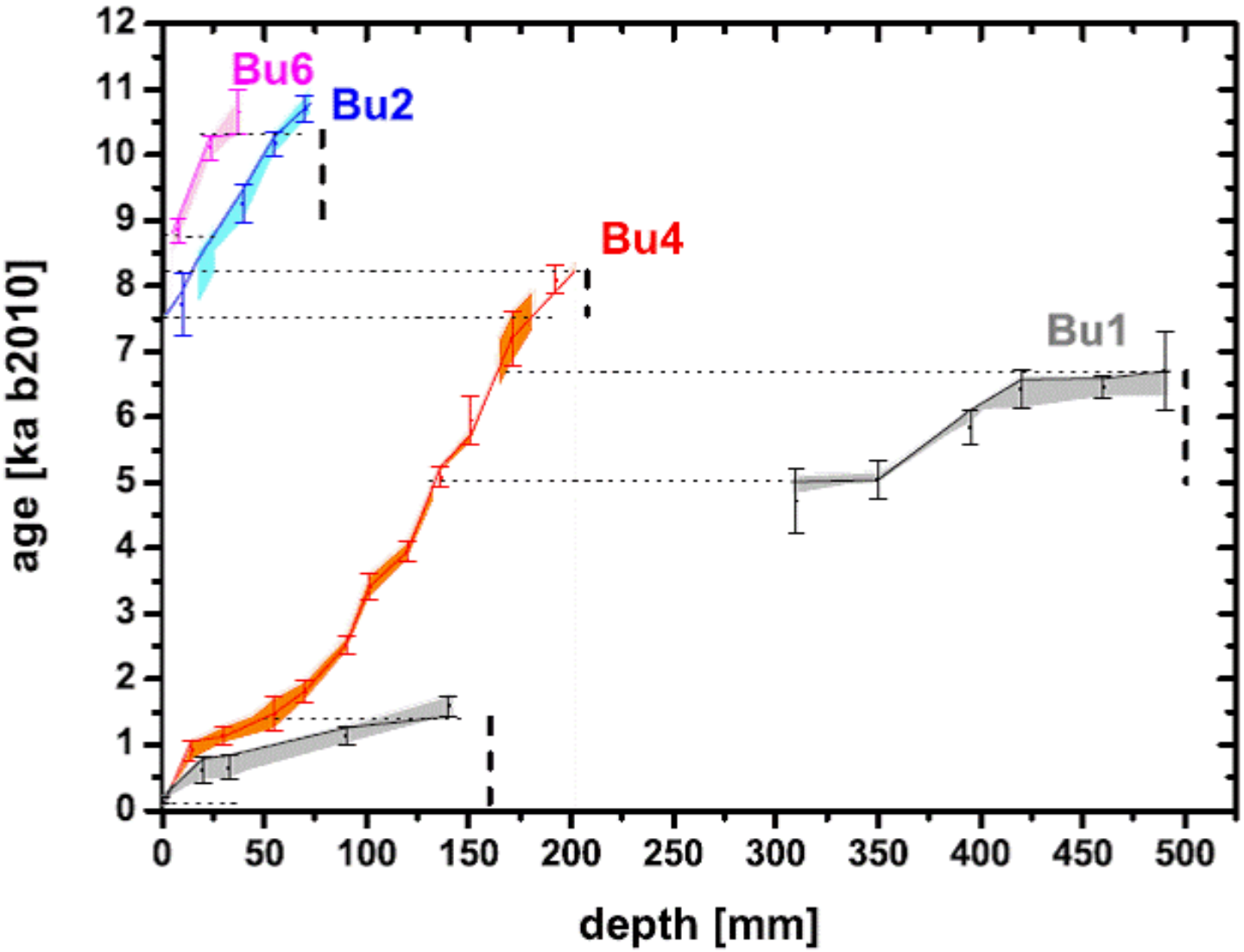
Analogy: density estimation



4.3 Nonparametric regression or smoothing—Stalagmite example

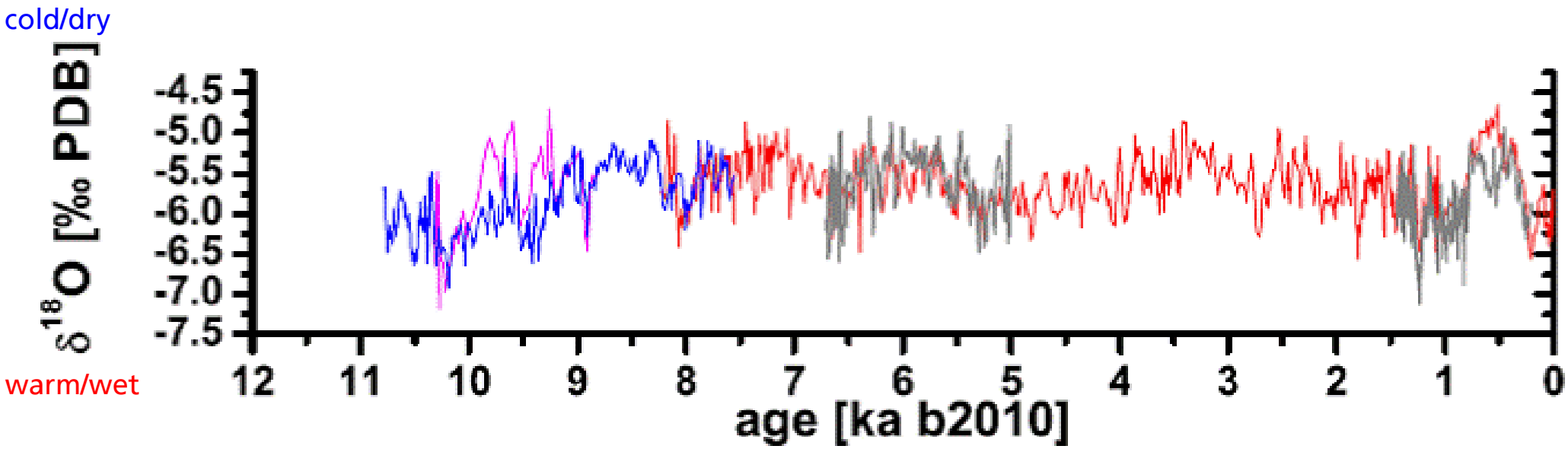


4.3 Nonparametric regression or smoothing—Stalagmite example



Fohlmeister et al. (in prep.) [ISCAM age model]

4.3 Nonparametric regression or smoothing—Stalagmite example



Oxygen isotope proxy

winter temperature

winter precipitation

Fohlmeister et al. (in prep.) [ISCAM age model]

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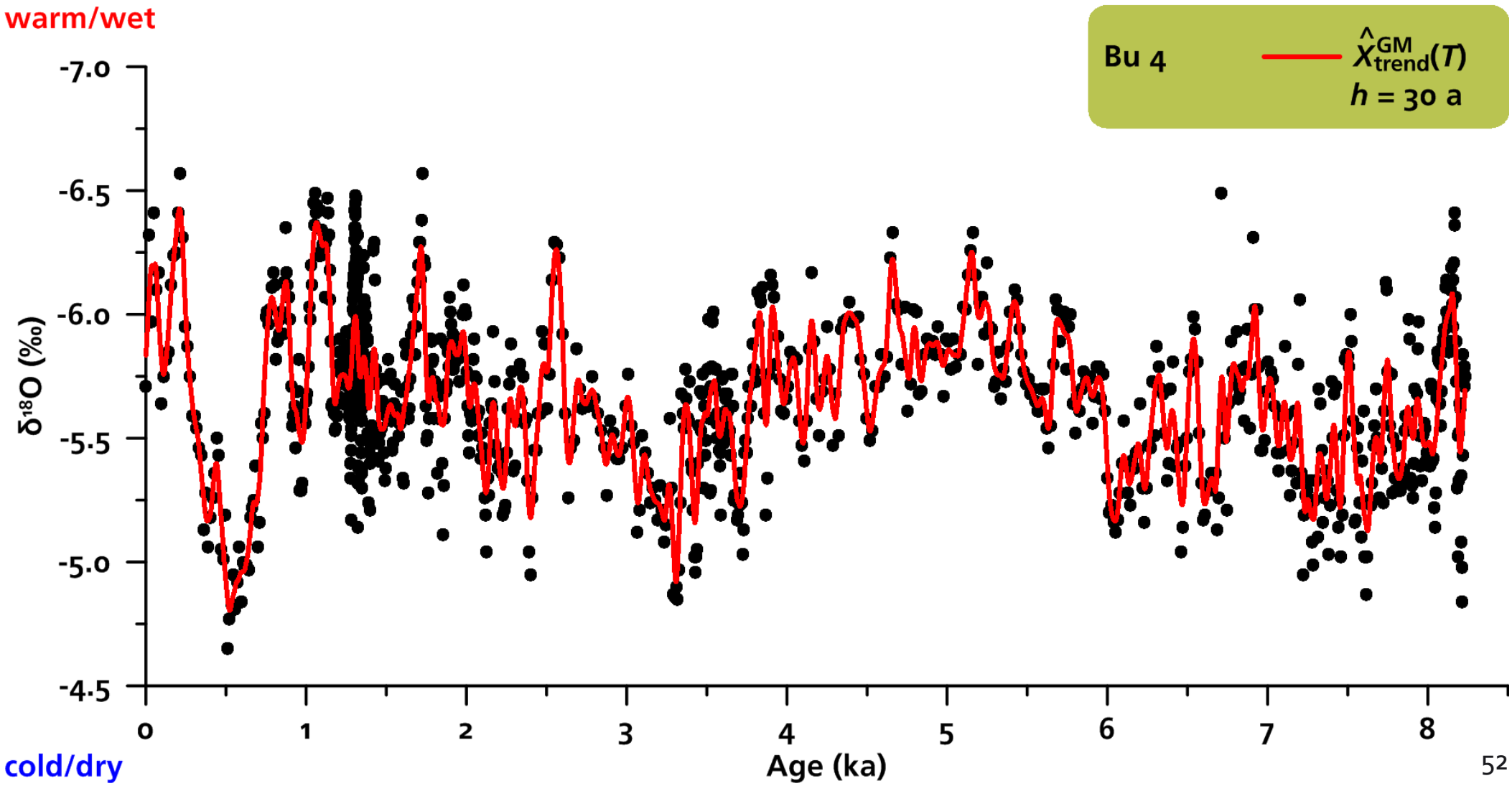

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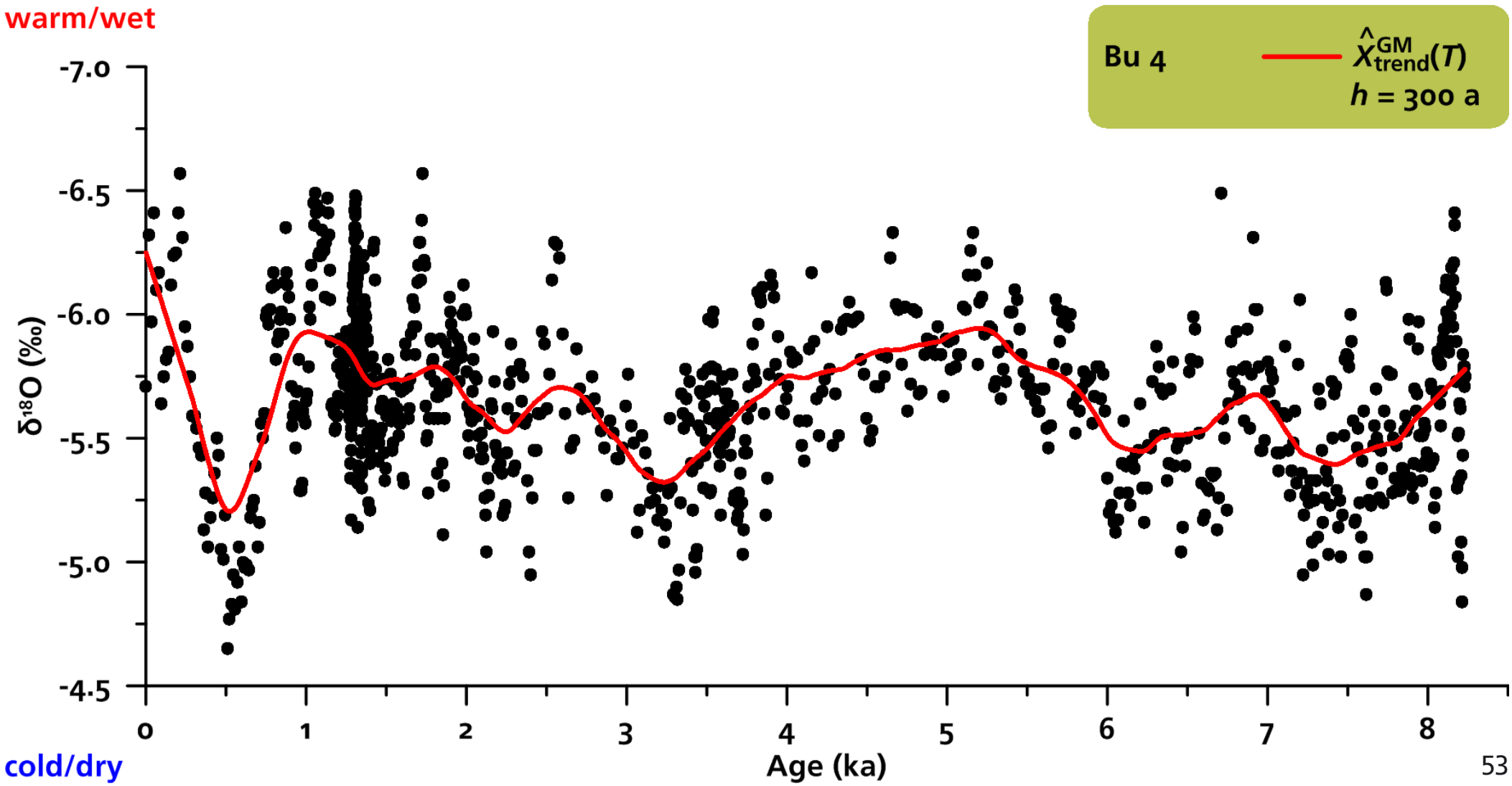

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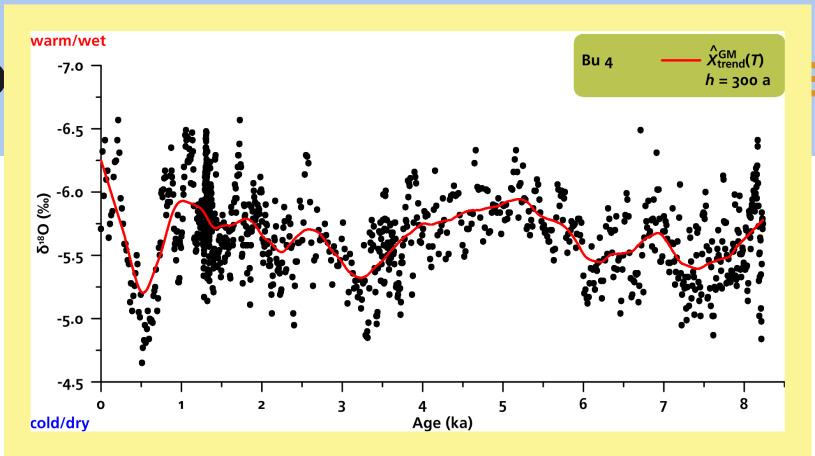
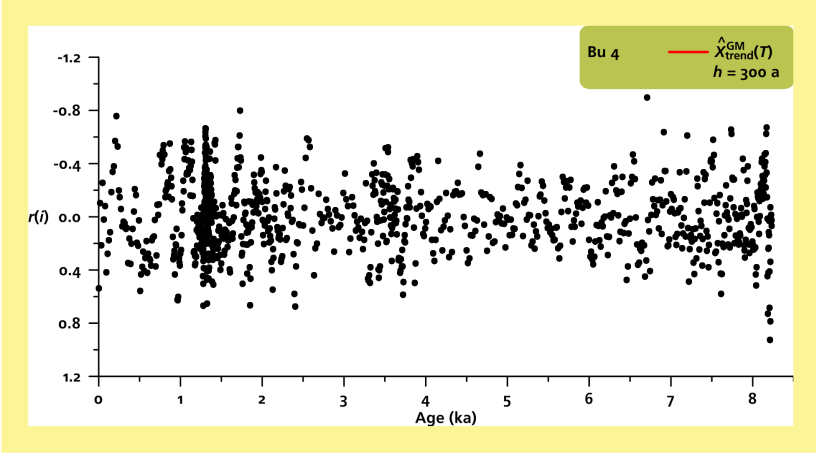
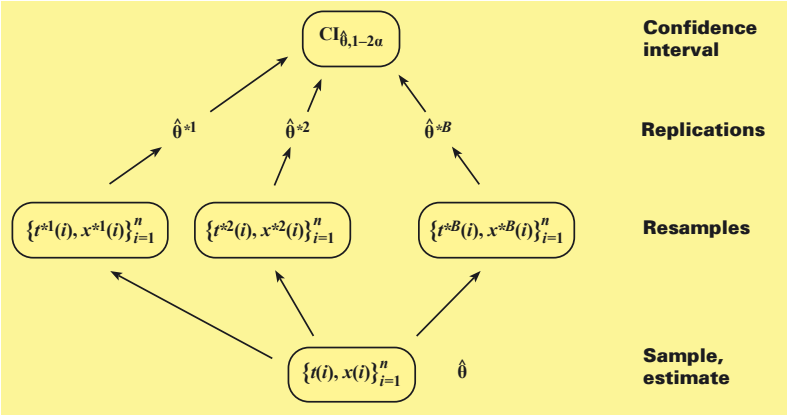
4.3 Nonparametric regression or smoo

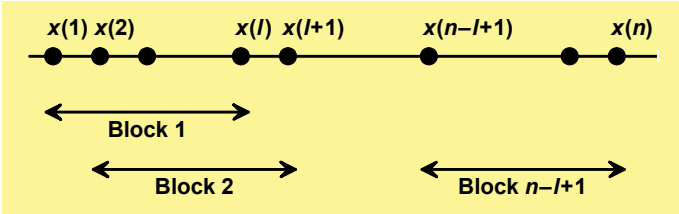


Bootstrap error band (pointwise)

- Take fit curve
- Add blocks of residuals†
- Resample: $\{t(i), x^*(i)\}$
- Re-estimate: replication
- Repeat many times
- For each T: error bar (replications)
- Band: connect error bars over T







Book, Algorithm 3.1, Fig. 3.3, Section 4.2.2.3 [† used here actually not MBB but ARB (Algorithm 3.5)]

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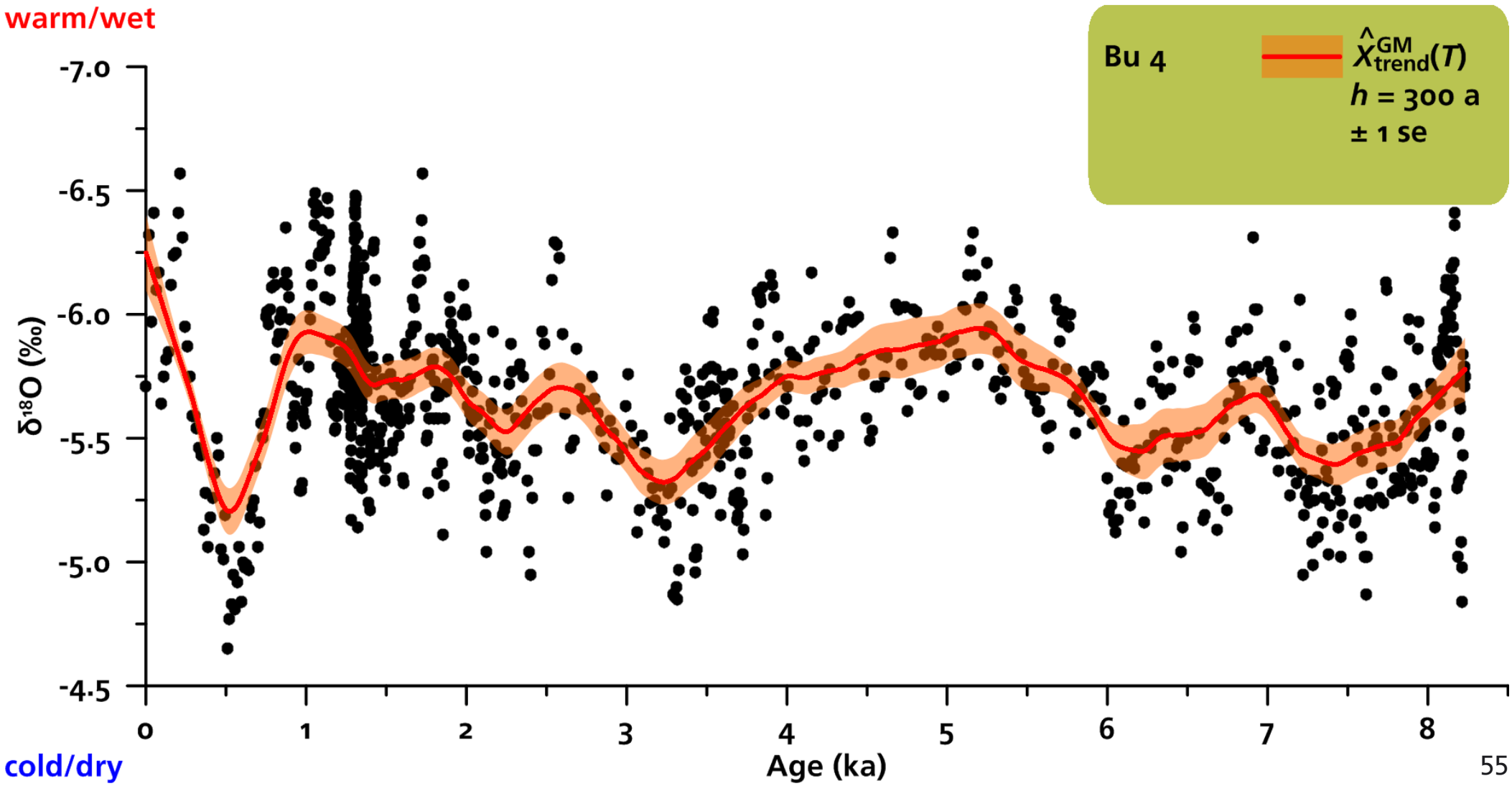

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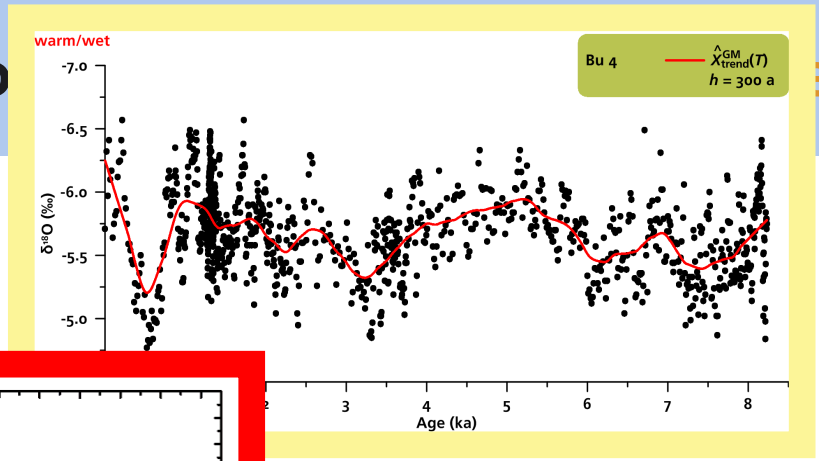

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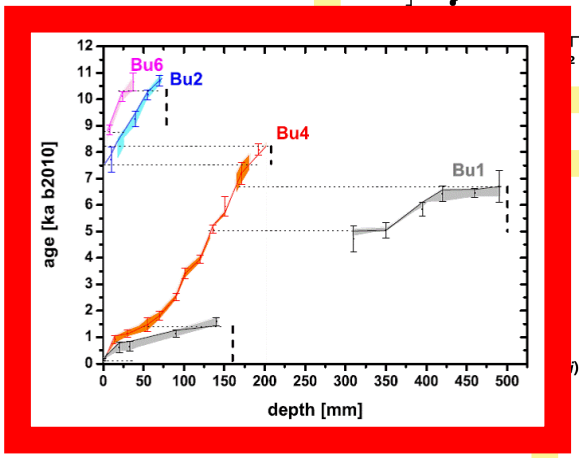

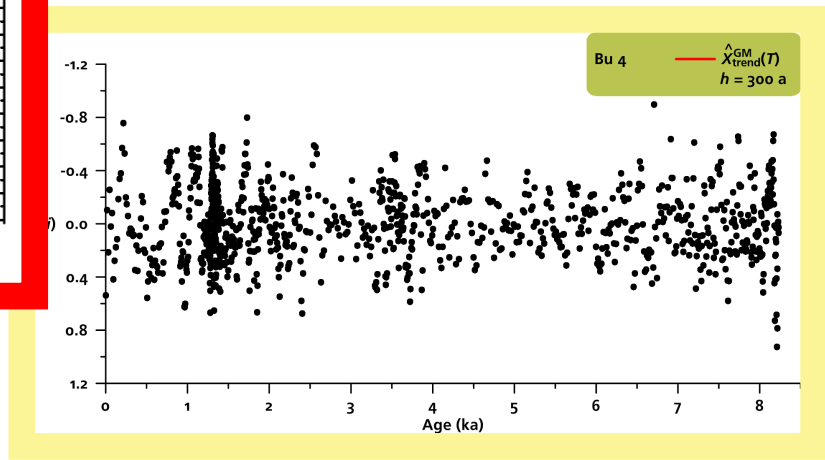

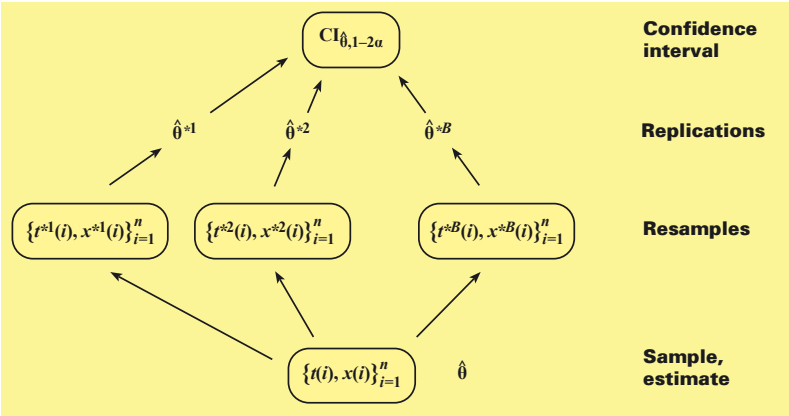

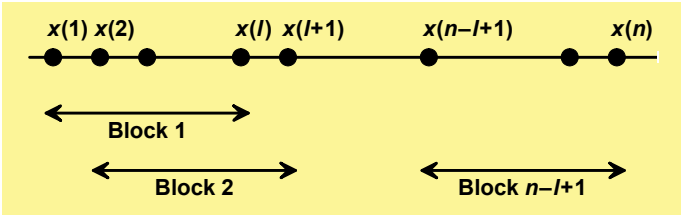

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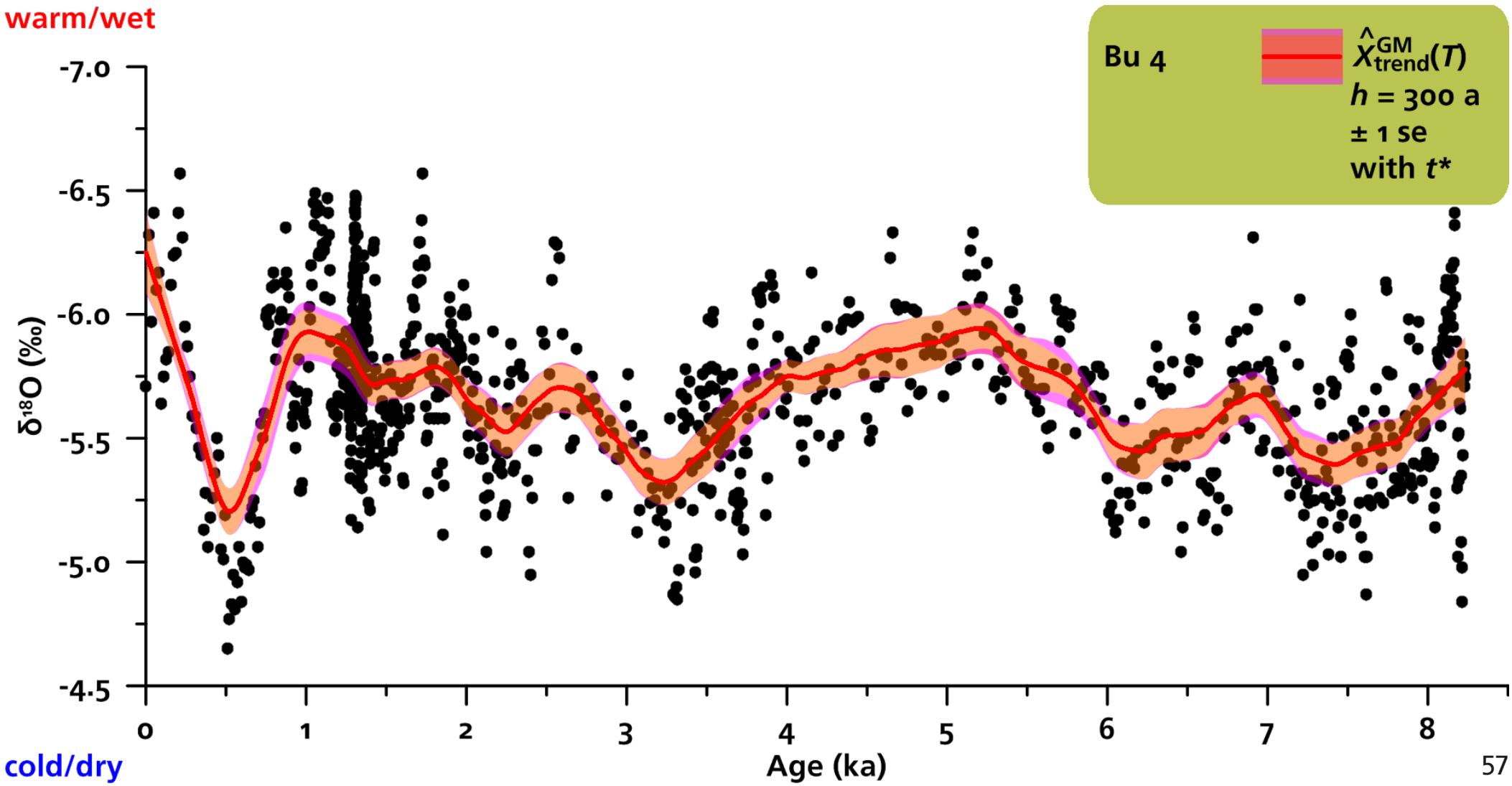

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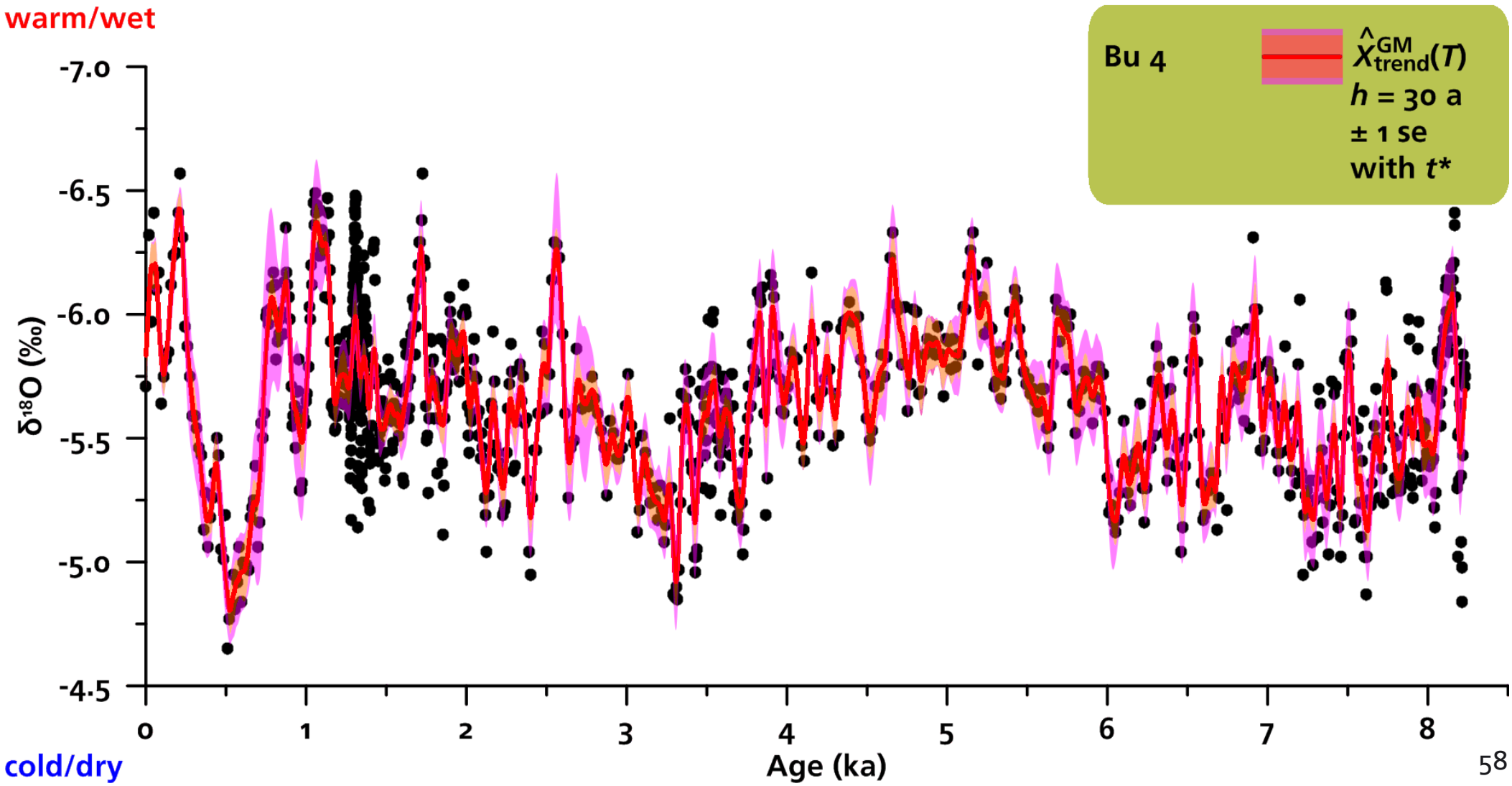

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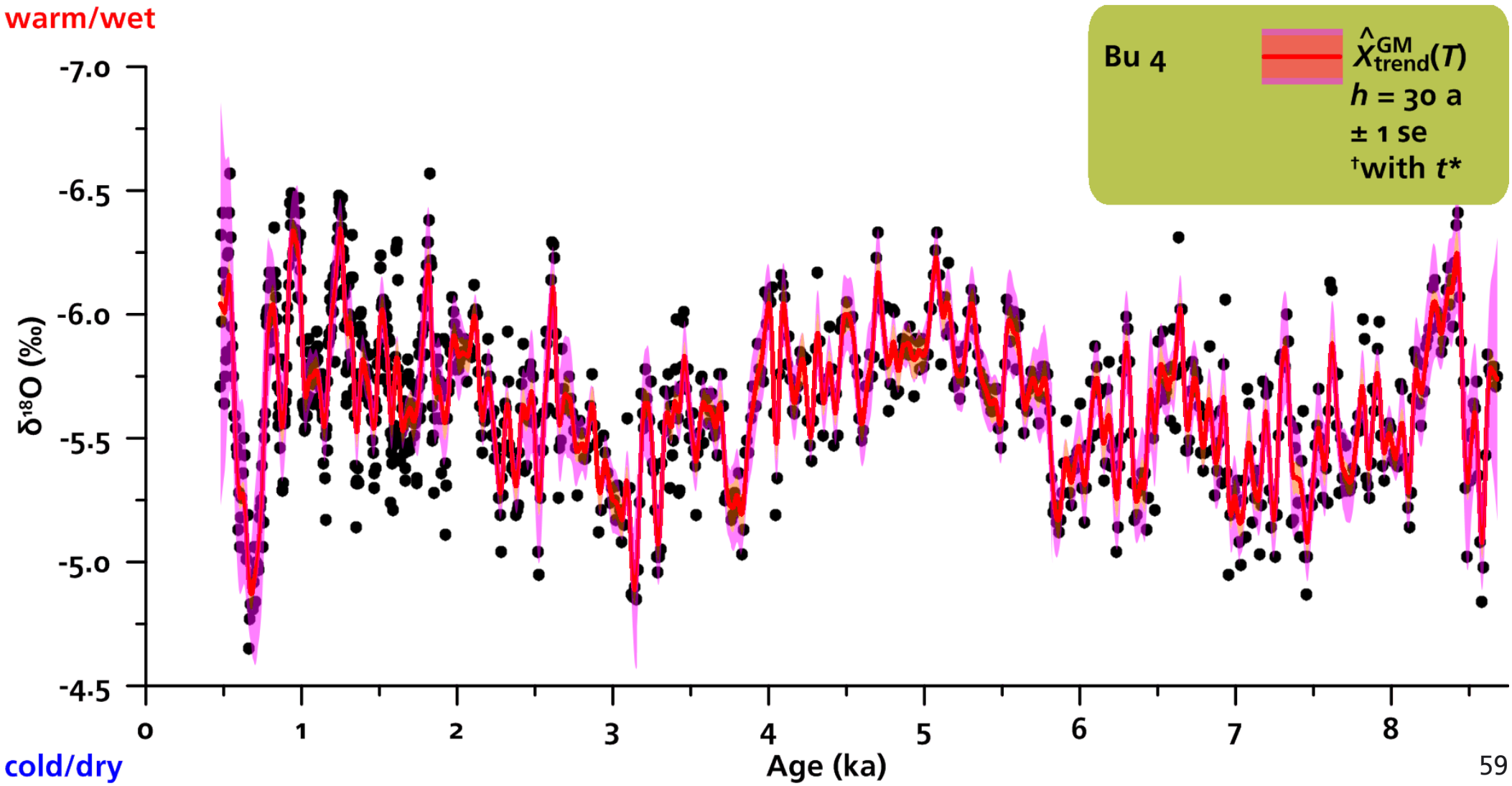

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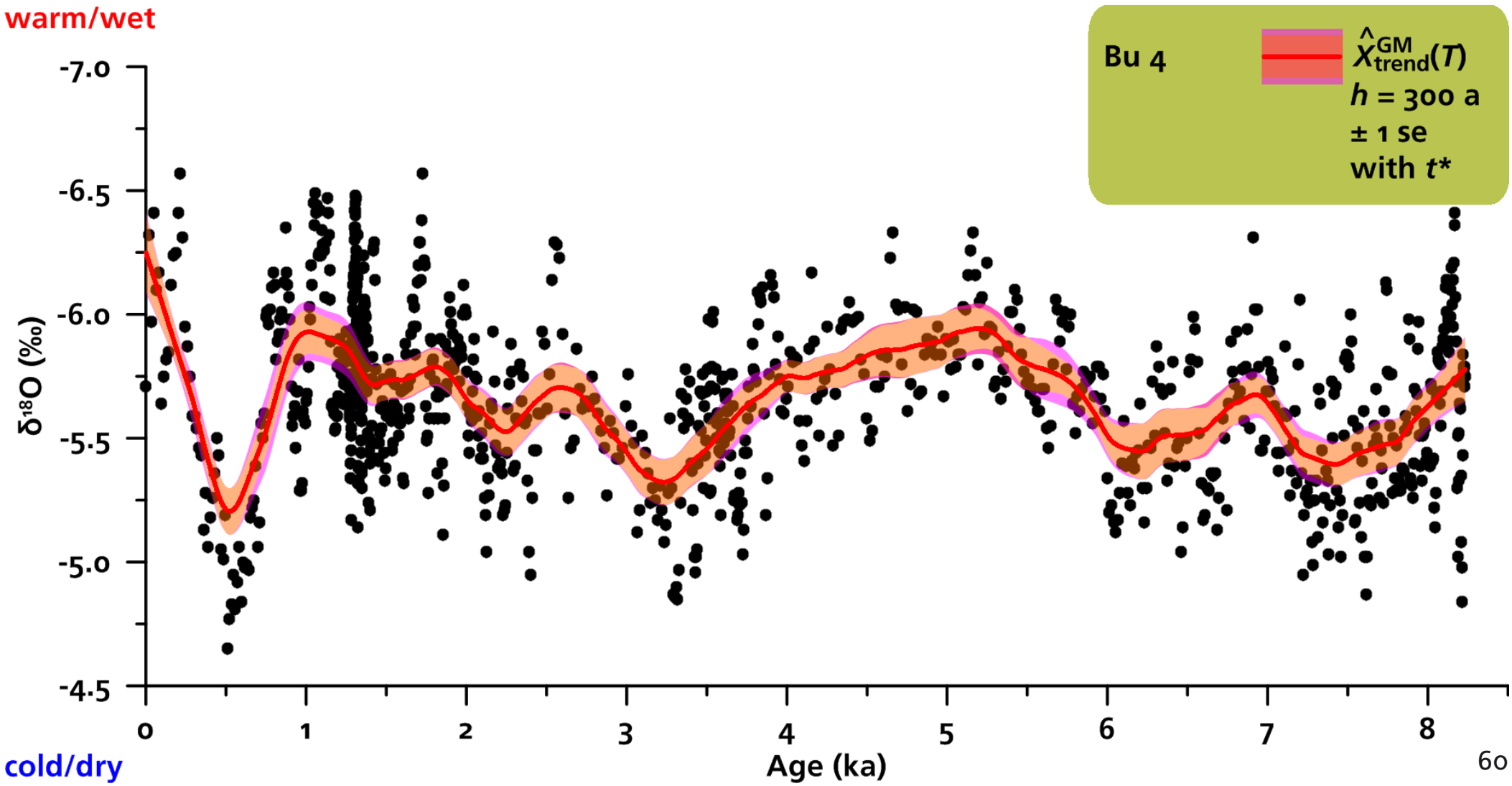
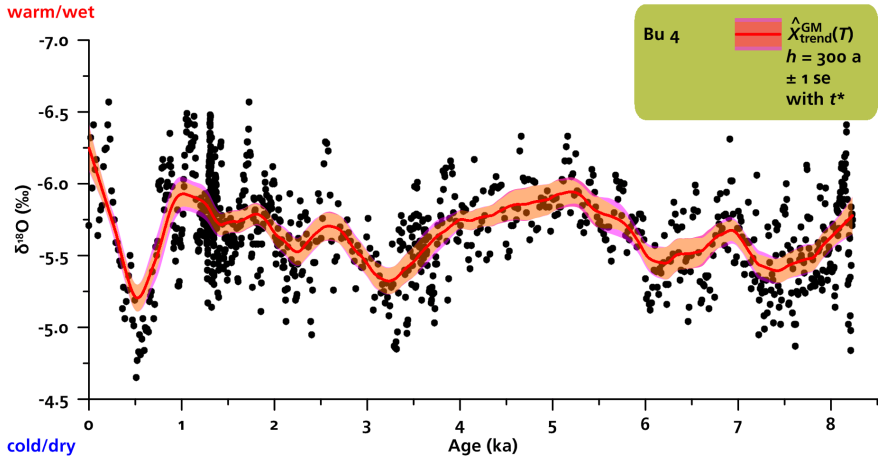


Mudelsee et al. (in prep.) [ISCAM age model]

4.3 Nonparametric regression or smoothing—Stalagmite example

Next steps

- Discuss age models
  - ISCAM, STALAGE
    - agreement/deviations
    - accuracy
- Discuss bandwidth selection
  - Physical arguments (e.g., solar cycles)
- Monte Carlo experiments
  - Coverage performance of error bands



Mudelsee et al. (in prep.)

Chapters

$$X(T) = X_{\text{trend}}(T) + X_{\text{out}}(T) + S(T) \cdot X_{\text{noise}}(T), \qquad (1.1)$$

Climate Time Series Analysis: Conclusions

(1) $\hat{\theta} \neq \theta$ and importance of computational techniques (bootstrap) to obtain realistic error bars.

(2) Paleo challenges: uneven spacing
uncertain timescales

Climate Time Series Analysis: Conclusions

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Thanks!

Postdoc job: www.climate-risk-analysis.com

4.3 Nonparametric regression or smoothing

$$\widehat{X}^{\mathrm{GM}}_{\mathrm{trend}}(T) = h^{-1} \sum_{i=1}^{n} \left[\int_{s(i-1)}^{s(i)} K\left(\frac{T-y}{h}\right) dy \right] X(i), \qquad (4.53)$$

where $T(i-1) \leq s(i-1) \leq T(i)$ (e.g., $s(i-1) = [T(i-1) + T(i)]/2$ with $s(0)$ and $s(n)$ being the upper and lower bounds of the T interval, respectively).

$$\widehat{X}^{\mathrm{PC}}_{\mathrm{trend}}(T) = h^{-1} \sum_{i=1}^{n} [T(i) - T(i-1)] \; K\left[\frac{T-T(i)}{h}\right] X(i), \qquad (4.52)$$

Kernel smoothing

- K, kernel function (: K = 0 outside, K = const. > 0 inside)
- h, bandwidth

Priestley and Chao (1972), Gasser and Müller (1979, 1984)

Bootstrap resampling

Autoregressive bootstrap (ARB)

- Take noise residuals *r*(*i*)
- Fit AR(1) model
- Take white-noise residuals $\epsilon(i) = [r(i) - \widehat{a}' \cdot r(i-1)]$
- Draw randomly from *e*(*i*) with replacement (ordinary bootstrap)
- Plug in resampled *e*(*i*)
- Plug in resampled *r*(*i*)